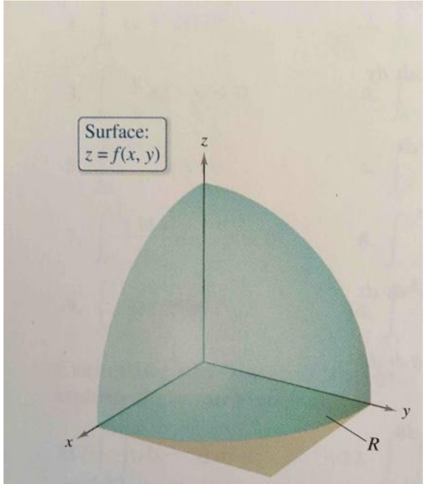
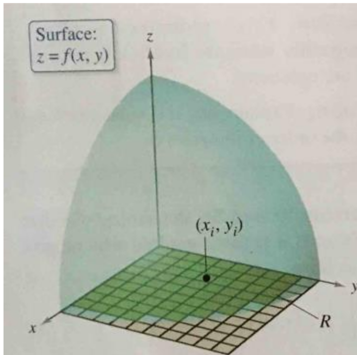
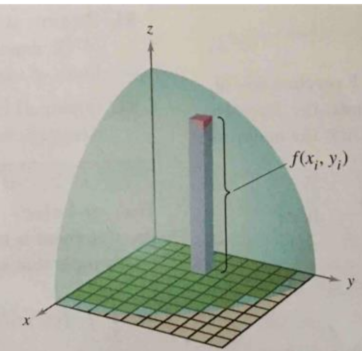


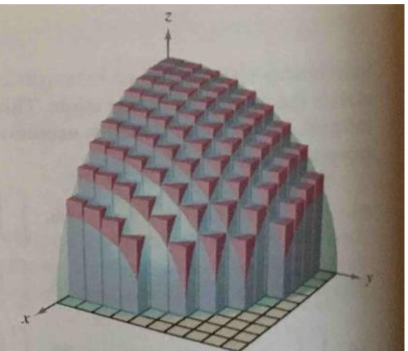
Figure 14.8



The rectangles lying within R form an inner partition of R.
Figure 14.9



Rectangular prism whose base has an area of ΔA_i and whose height is $f(x_i, y_i)$
Figure 14.10



Volume approximated by rectangular prisms
Figure 14.11

Definition of Double Integral

If f is defined on a closed, bounded region R in the xy-plane, then the **double integral of f over R** is

$$\int_R\int f(x, y)\, dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

Volume of a Solid Region

If f is integrable over a plane region R and $f(x, y) \geq 0$ for all (x, y) in R, then the volume of the solid region that lies above R and below the graph of f is

$$V = \int_R\int f(x, y)\, dA.$$

THEOREM 14.1 Properties of Double Integrals

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1. $\int_R\int cf(x, y)\, dA = c\int_R\int f(x, y)\, dA$
2. $\int_R\int [f(x, y) \pm g(x, y)]\, dA = \int_R\int f(x, y)\, dA \pm \int_R\int g(x, y)\, dA$
3. $\int_R\int f(x, y)\, dA \geq 0, \quad \text{if } f(x, y) \geq 0$
4. $\int_R\int f(x, y)\, dA \geq \int_R\int g(x, y)\, dA, \quad \text{if } f(x, y) \geq g(x, y)$
5. $\int_R\int f(x, y)\, dA = \int_{R_1}\int f(x, y)\, dA + \int_{R_2}\int f(x, y)\, dA$, where R is the union of two nonoverlapping subregions R_1 and R_2.

THEOREM 14.2 Fubini's Theorem

Let f be continuous on a plane region R.

1. If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then

$$\int_R\int f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then

$$\int_R\int f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

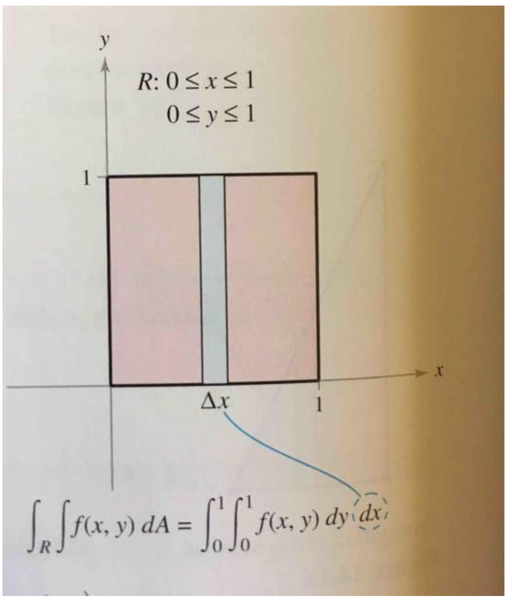
EXAMPLE 2 **Evaluating a Double Integral**

Evaluate

$$\int_R\int\left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right)dA$$

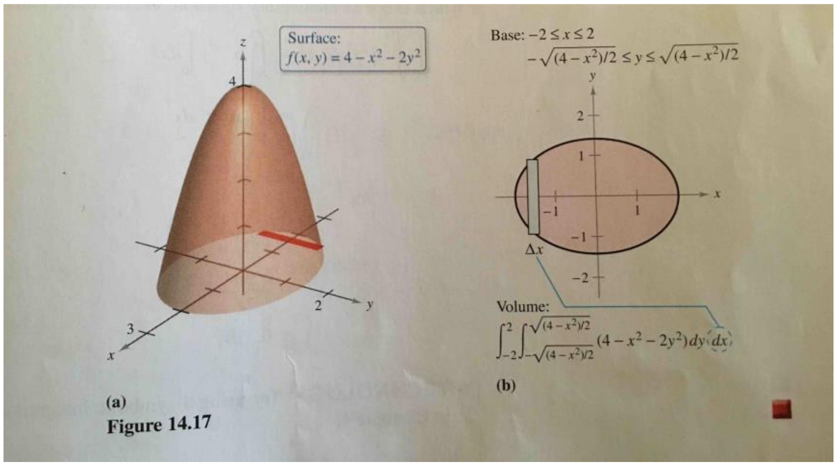
where R is the region given by

$$0 \le x \le 1, \quad 0 \le y \le 1.$$



EXAMPLE 3 Finding Volume by a Double Integral

Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane, as shown in Figure 14.17(a).



(a)
Figure 14.17

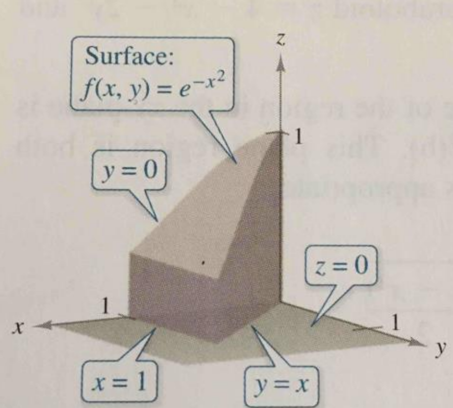
EXAMPLE 4 **Comparing Different Orders of Integration**

See LarsonCalculus.com for an interactive version of this type of example.

Find the volume of the solid region bounded by the surface

$$f(x, y) = e^{-x^2} \qquad \text{Surface}$$

and the planes $z = 0$, $y = 0$, $y = x$, and $x = 1$, as shown in Figure 14.18.



Base is bounded by $y = 0$, $y = x$, and $x = 1$.

Figure 14.18

EXAMPLE 5 Volume of a Region Bounded by Two Surfaces

Find the volume of the solid region bounded above by the paraboloid

$$z = 1 - x^2 - y^2 \qquad \text{Paraboloid}$$

and below by the plane

$$z = 1 - y \qquad \text{Plane}$$

as shown in Figure 14.20.

Average Value of a Function

Recall from Section 4.4 that for a function f in one variable, the average value of f on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Given a function f in two variables, you can find the average value of f over the plane region R as shown in the following definition.

Definition of the Average Value of a Function Over a Region

If f is integrable over the plane region R, then the **average value** of f over R is

$$\text{Average value} = \frac{1}{A}\int_R\int f(x, y)\,dA$$

where A is the area of R.

EXAMPLE 6 Finding the Average Value of a Function

Find the average value of

$$f(x, y) = \frac{1}{2}xy$$

over the plane region R, where R is a rectangle with vertices

$(0, 0)$, $(4, 0)$, $(4, 3)$, and $(0, 3)$.