Infinite Series

One important application of infinite sequences is in representing "infinite summations." Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \qquad \text{Infinite Series}$$

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series **diverges.**

EXAMPLE 1 Convergent and Divergent Series

a. The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

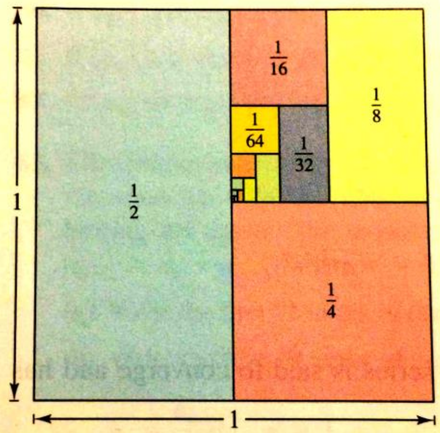
has the partial sums listed below. (You can also determine the p
series geometrically, as shown in Figure 9.6.)

b. The nth partial sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

c. The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$



You can determine the partial sums of the series in Example 1(a) geometrically using this figure.

Figure 9.6

The series in Example 1(b) is a **telescoping series** of the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdot \cdot \cdot . \qquad \text{Telescoping series}$$

EXAMPLE 2 Writing a Series in Telescoping Form

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Geometric Series

The series in Example 1(a) is a **geometric series.** In general, the series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdot \cdot \cdot + ar^n + \cdot \cdot \cdot, \quad a \neq 0$$

Geo

is a **geometric series** with ratio r, $r \neq 0$.

THEOREM 9.6 Convergence of a Geometric Series

A geometric series with ratio r diverges when $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

EXAMPLE 3 Convergent and Divergent Geometric

a. The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$. Because $0 < |r| < 1$, the series co
sum is

b. The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

has a ratio of $r = \frac{3}{2}$. Because $|r| \geq 1$, the series diverges.

EXAMPLE 4 A Geometric Series for a Repeating D

See LarsonCalculus.com for an interactive version of this type of exa

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

THEOREM 9.7 Properties of Infinite Series

Let Σa_n and Σb_n be convergent series, and let A, B, and c be real numbers. If $\Sigma a_n = A$ and $\Sigma b_n = B$, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} ca_n = cA$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

THEOREM 9.8 Limit of the *n*th Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

THEOREM 9.9 *n*th-Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE 5 Using the *n*th-Term Test

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have

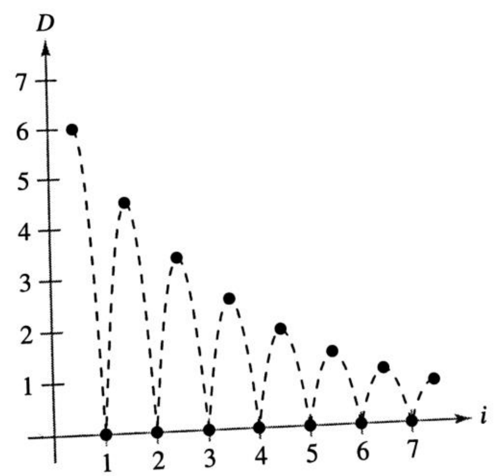
b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$, you have

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

s in

EXAMPLE 6 **Bouncing Ball Problem**

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



The height of each bounce is three-fourths the height of the preceding bounce.

Figure 9.7