Definition of Unit Tangent Vector

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. The **unit tangent vector** $\mathbf{T}(t)$ at t is defined as

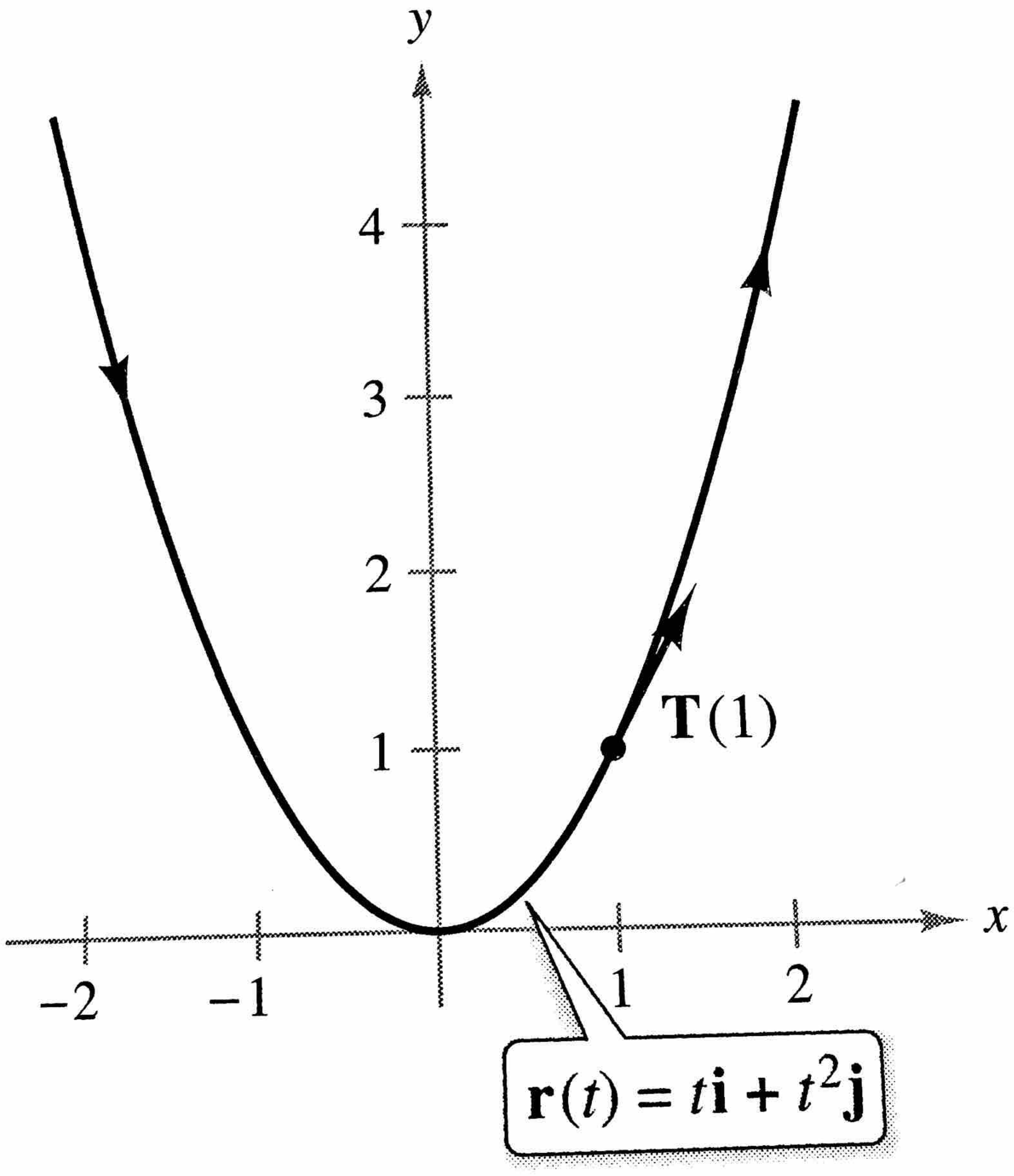
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{r}'(t) \neq \mathbf{0}.$$

EXAMPLE 1 **Finding the Unit Tangent Vector**

Find the unit tangent vector to the curve given by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

when $t = 1$.



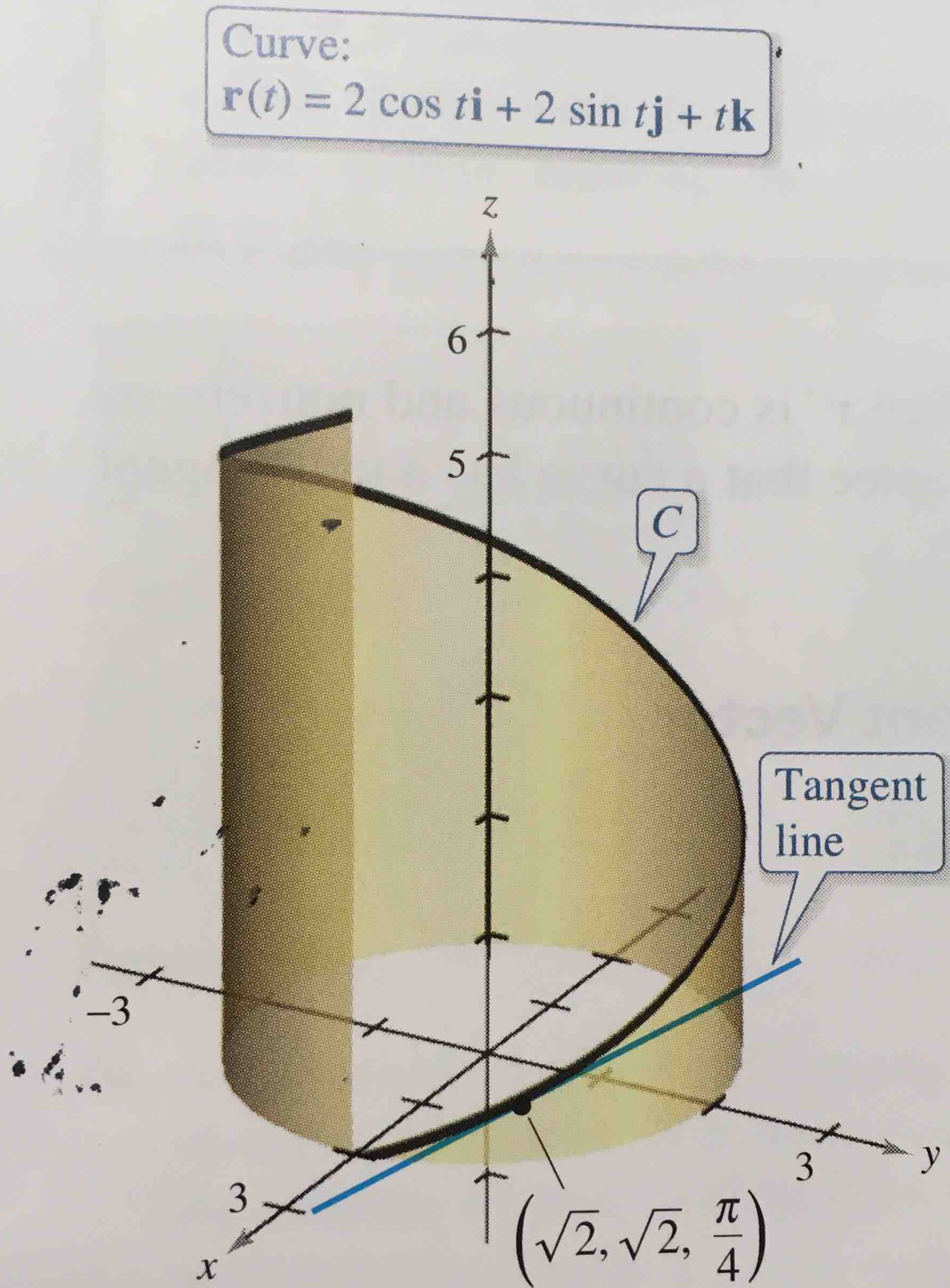
The direction of the unit tangent vector depends on the orientation of the curve.
Figure 12.20

EXAMPLE 2 Finding the Tangent Line at a Point on a Curve

Find $\mathbf{T}(t)$ and then find a set of parametric equations for the tangent line to the helix given by

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$$

at the point $\left(\sqrt{2}, \sqrt{2}, \frac{\pi}{4}\right)$.



The tangent line to a curve at a point is determined by the unit tangent vector at the point.

Figure 12.21

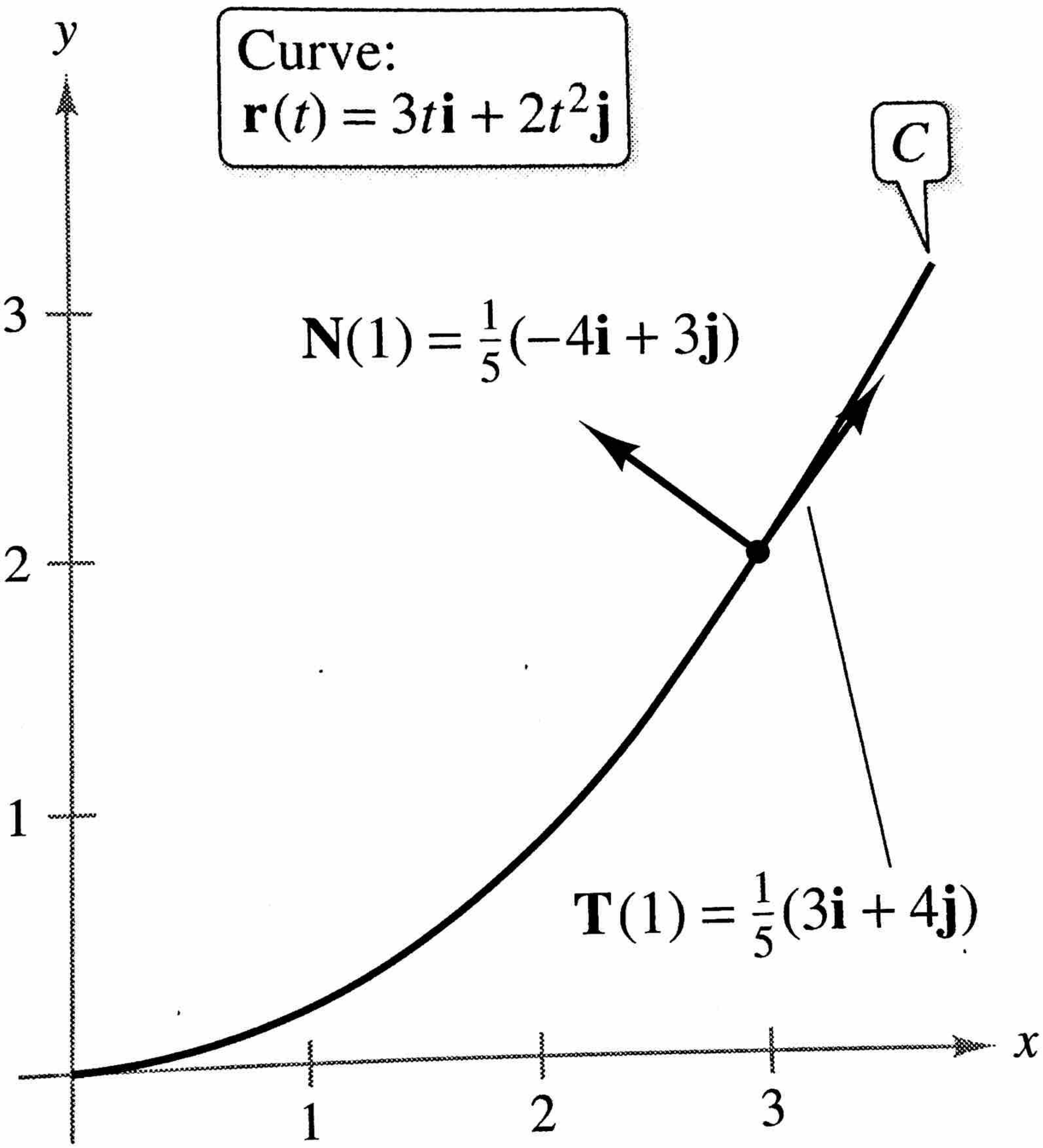
Definition of Principal Unit Normal Vector

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the **principal unit normal vector** at t is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

EXAMPLE 3 **Finding the Principal Unit Normal Vector**

Find $\mathbf{N}(t)$ and $\mathbf{N}(1)$ for the curve represented by $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}$.



The principal unit normal vector points toward the concave side of the curve.

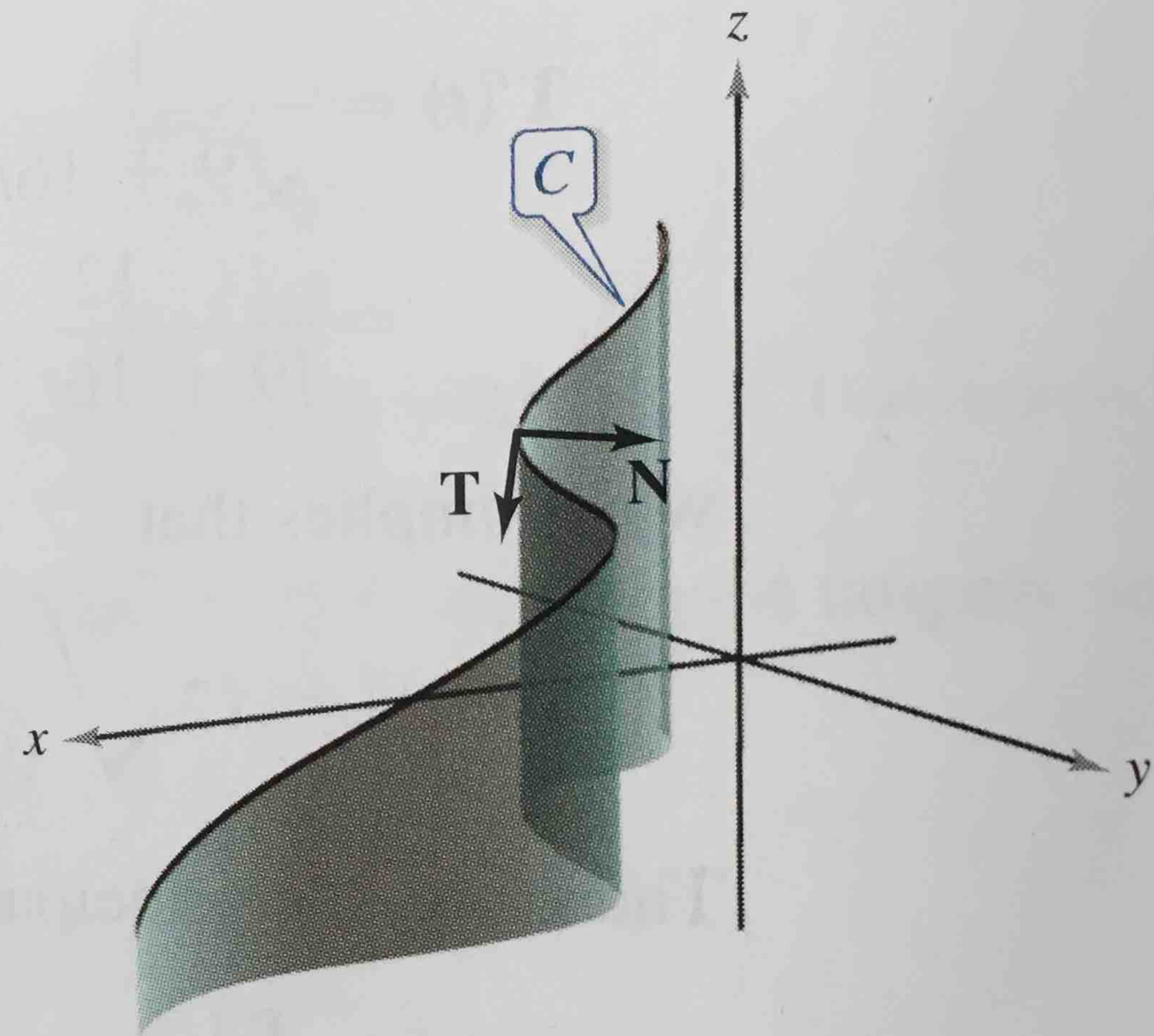
Figure 12.22

The principal unit normal vector can be difficult to evaluate algebraically. For plane curves, you can simplify the algebra by finding

$$\mathbf{T}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad \text{Unit tangent vector}$$

and observing that $\mathbf{N}(t)$ must be either

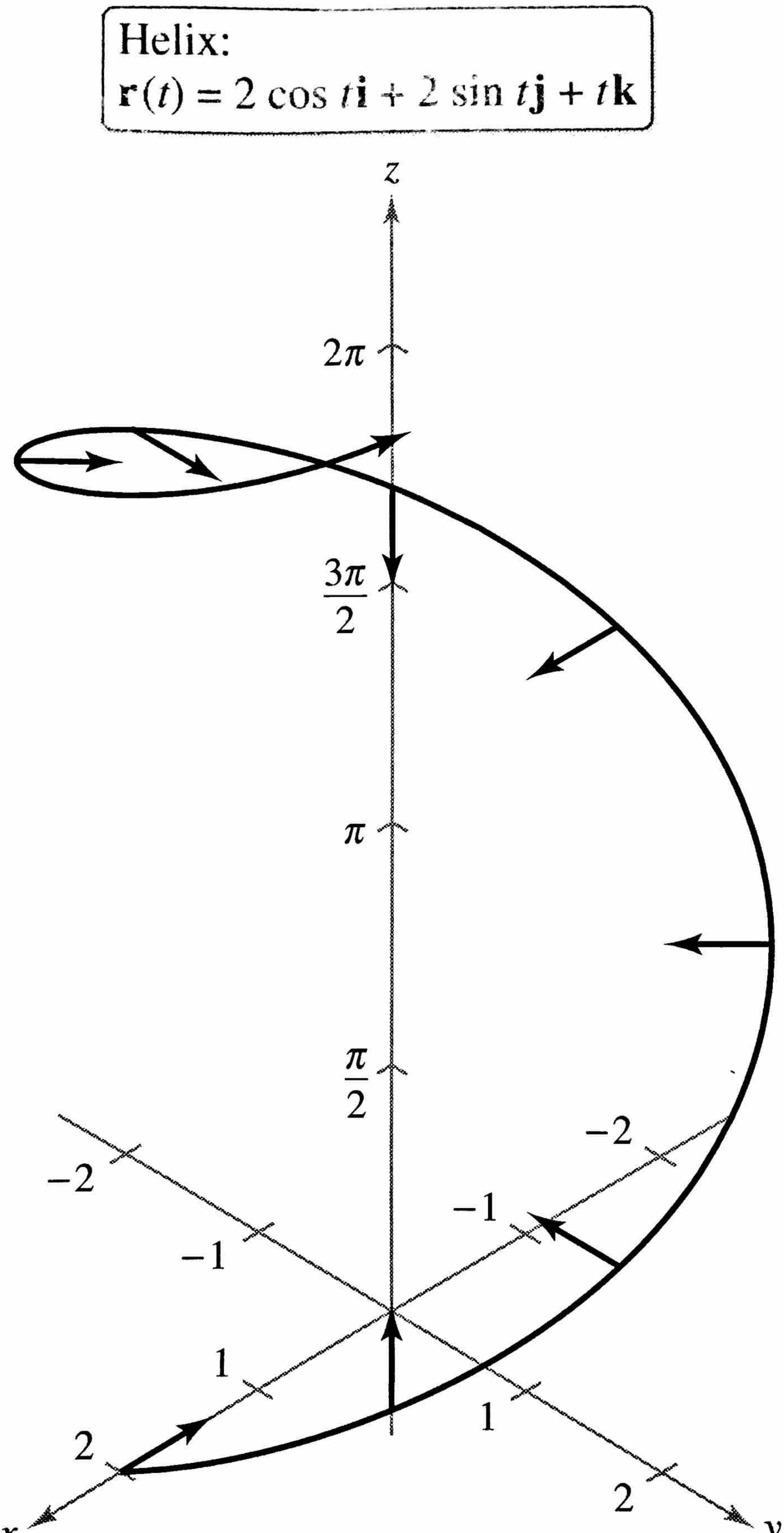
$$\mathbf{N}_1(t) = y(t)\mathbf{i} - x(t)\mathbf{j} \quad \text{or} \quad \mathbf{N}_2(t) = -y(t)\mathbf{i} + x(t)\mathbf{j}.$$



At any point on a curve, a unit normal vector is orthogonal to the unit tangent vector. The *principal* unit normal vector points in the direction in which the curve is turning.
Figure 12.23

EXAMPLE 4 Finding the Principal Unit Normal Vector

Find the principal unit normal vector for the helix $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$.



$\mathbf{N}(t)$ is horizontal and points toward the z-axis.

Figure 12.24

THEOREM 12.4 Acceleration Vector

If $\mathbf{r}(t)$ is the position vector for a smooth curve C and $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t)$ lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

$$\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t).$$

The next theorem lists some convenient formulas for $a_{\mathbf{N}}$ and $a_{\mathbf{T}}$.