$$\cos^2\theta = 1 - \sin^2\theta$$
$$\sec^2\theta = 1 + \tan^2\theta$$
$$\tan^2\theta = \sec^2\theta - 1$$

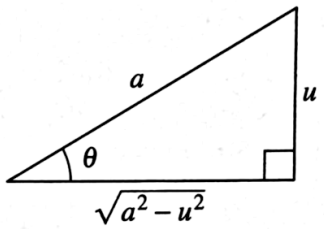
Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$u = a \sin\theta$.

Then $\sqrt{a^2 - u^2} = a\cos\theta$, where

$-\pi/2 \le \theta \le \pi/2$.

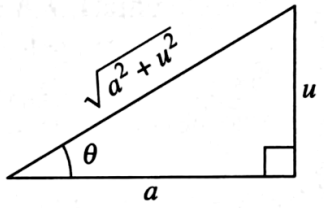


2. For integrals involving $\sqrt{a^2 + u^2}$, let

$u = a \tan\theta$.

Then $\sqrt{a^2 + u^2} = a\sec\theta$, where

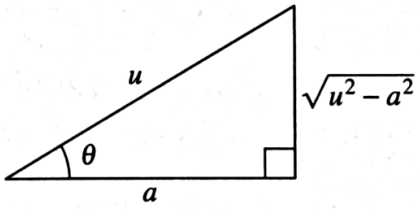
$-\pi/2 < \theta < \pi/2$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let

$u = a \sec\theta$.

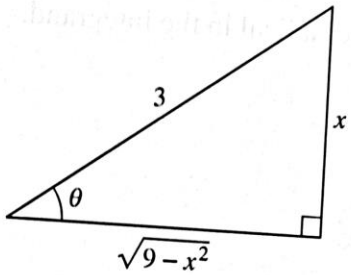
Then



$$\sqrt{u^2 - a^2} = \begin{cases} a\tan\theta \text{ for } u > a, \text{ where } 0 \le \theta \le \pi/2 \\ -a\tan\theta \text{ for } u < -a, \text{ where } \pi/2 < \theta \le \pi. \end{cases}$$

EXAMPLE 1 **Trigonometric Substitution: $u = a \sin \theta$**

Find $\int \frac{dx}{x^2\sqrt{9 - x^2}}$.

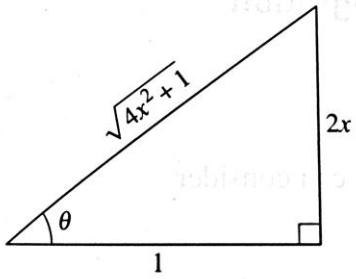


$\sin \theta = \frac{x}{3}$, $\cot \theta = \frac{\sqrt{9 - x^2}}{x}$

Figure 8.6

EXAMPLE 2 **Trigonometric Substitution: $u = a \tan \theta$**

Find $\int \frac{dx}{\sqrt{4x^2 + 1}}$.



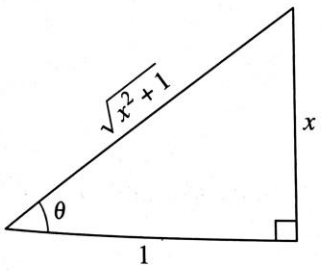
$\tan \theta = 2x$, $\sec \theta = \sqrt{4x^2 + 1}$

Figure 8.7

EXAMPLE 3 **Trigonometric Substitution: Rational Powers**

See LarsonCalculus.com for an interactive version of this type of example.

Find $\int \frac{dx}{(x^2 + 1)^{3/2}}$.

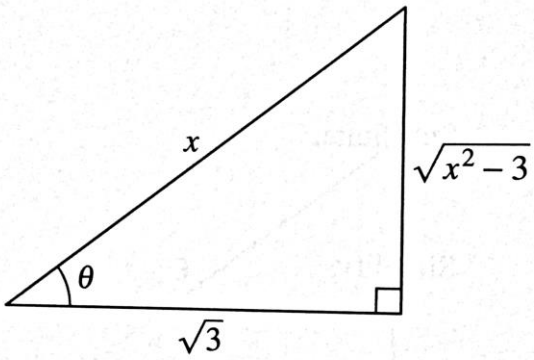


$\tan \theta = x$, $\sin \theta = \frac{x}{\sqrt{x^2 + 1}}$

Figure 8.8

EXAMPLE 4 **Converting the Limits of Integration**

Evaluate $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx.$



$\sec\theta = \frac{x}{\sqrt{3}}$, $\tan\theta = \frac{\sqrt{x^2-3}}{\sqrt{3}}$

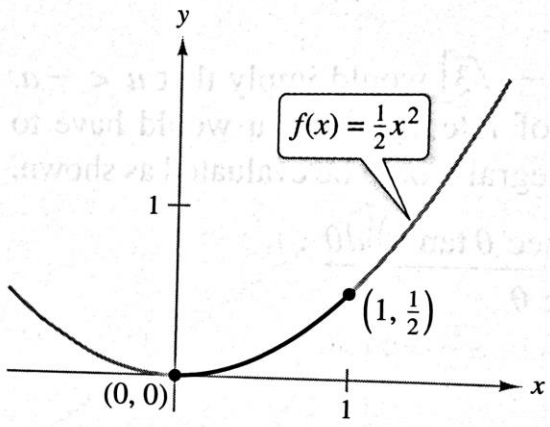
Figure 8.9

THEOREM 8.2 Special Integration Formulas ($a > 0$)

1. $\int \sqrt{a^2 - u^2}\, du = \frac{1}{2}\left(a^2 \arcsin\frac{u}{a} + u\sqrt{a^2 - u^2}\right) + C$
2. $\int \sqrt{u^2 - a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln\left|u + \sqrt{u^2 - a^2}\right|\right) + C, \quad u > a$
3. $\int \sqrt{u^2 + a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 + a^2} + a^2 \ln\left|u + \sqrt{u^2 + a^2}\right|\right) + C$

EXAMPLE 5 Finding Arc Length

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$ (see Figure 8.10).

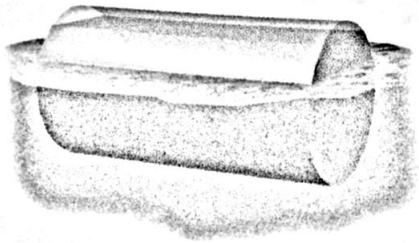


The arc length of the curve from $(0, 0)$ to $\left(1, \frac{1}{2}\right)$

Figure 8.10

EXAMPLE 6 **Comparing Two Fluid Forces**

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot), as shown in Figures 8.11 and 8.12. (The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 foot of the barrel is empty.) Compare the fluid forces against one end of the barrel from the inside and from the outside.



The barrel is not quite full of oil—the top 0.2 foot of the barrel is empty.
Figure 8.11

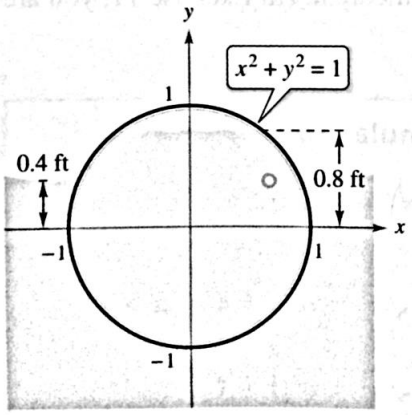


Figure 8.12