Definition of the Derivative of a Vector-Valued Function

The **derivative of a vector-valued function r** is

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(t)$ exists, then $\mathbf{r}$ is **differentiable at t.** If $\mathbf{r}'(t)$ exists for all t in an open interval I, then $\mathbf{r}$ is **differentiable on the interval I.** Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

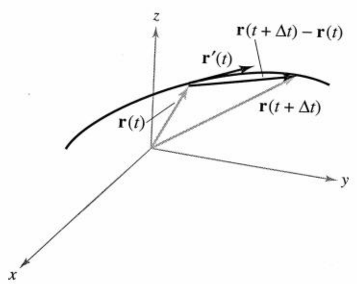


Figure 12.8

THEOREM 12.1 Differentiation of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of t, then

   $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$. Plane

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then

   $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$. Space

Differentiation of a Vector-Valued Function

See LarsonCalculus.com for an interactive version of this type of example.

For the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 2)\mathbf{j}$$

find $\mathbf{r}'(t)$. Then sketch the plane curve represented by $\mathbf{r}(t)$ and the graphs of $\mathbf{r}(1)$ and $\mathbf{r}'(1)$.

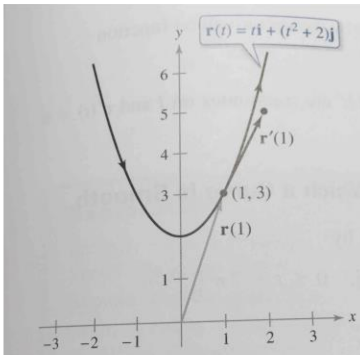


Figure 12.9

EXAMPLE 2 **Higher-Order Differentiation**

For the vector-valued function

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$$

find each of the following.

a. $\mathbf{r}'(t)$

b. $\mathbf{r}''(t)$

c. $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$

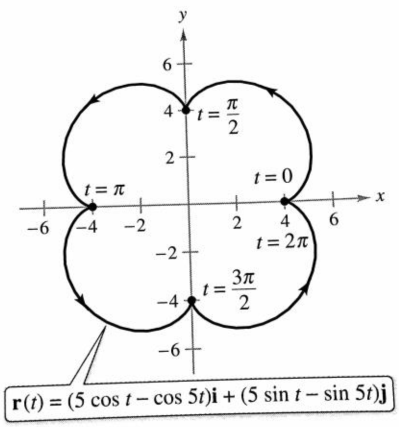
d. $\mathbf{r}'(t) \times \mathbf{r}''(t)$

EXAMPLE 3 **Finding Intervals on Which a Curve Is Smooth**

Find the intervals on which the epicycloid C given by

$$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}, \quad 0 \le t \le 2\pi$$

is smooth.



The epicycloid is not smooth at the points where it intersects the axes.
Figure 12.10

THEOREM 12.2 Properties of the Derivative

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let w be a differentiable real-valued function of t, and let c be a scalar.

1. $\dfrac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
2. $\dfrac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $\dfrac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$
4. $\dfrac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $\dfrac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

EXAMPLE 4 **Using Properties of the Derivative**

For $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}$ and $\mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$, find

a. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and **b.** $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)]$.

Definition of Integration of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of **r** is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} \qquad \text{Plane}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}.$$

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of **r** is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k} \qquad \text{Space}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$

EXAMPLE 5 Integrating a Vector-Valued Function

Find the indefinite integral

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt.$$

EXAMPLE 6 Definite Integral of a Vector-Valued Function

Evaluate the integral

$$\int_0^1 \mathbf{r}(t)\,dt = \int_0^1 \left(\sqrt[3]{t}\,\mathbf{i} + \frac{1}{t+1}\mathbf{j} + e^{-t}\,\mathbf{k} \right) dt.$$

EXAMPLE 7 **The Antiderivative of a Vector-Valued Function**

Find the antiderivative of

$$\mathbf{r}'(t) = \cos 2t\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$$

that satisfies the initial condition

$$\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$