Definition of a Function of Two Variables

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is a **function of x and y.** The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f. For the function

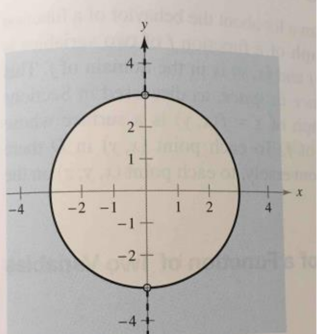
$$z = f(x, y)$$

x and y are called the **independent variables** and z is called the **dependent variable.**

EXAMPLE 1 **Domains of Functions of Several Variables**

Find the domain of each function.

a. $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$ **b.** $g(x, y, z) = \dfrac{x}{\sqrt{9 - x^2 - y^2 - z^2}}$



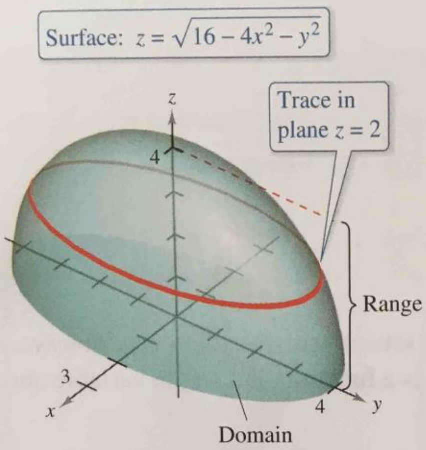
Domain of $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$

Figure 13.1

$(f \pm g)(x, y) = f(x, y) \pm g(x, y)$	Sum or difference
$(fg)(x, y) = f(x, y)g(x, y)$	Product
$\dfrac{f}{g}(x, y) = \dfrac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0$	Quotient

ot form the composite of two functions of several variables. You can, ho
composite function $(g \circ h)(x, y)$, where g is a function of a single variab
ction of two variables.

$(g \circ h)(x, y) = g(h(x, y))$	Composition



The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.
Figure 13.3

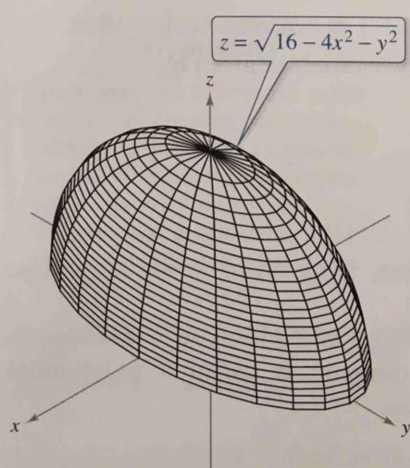


Figure 13.4

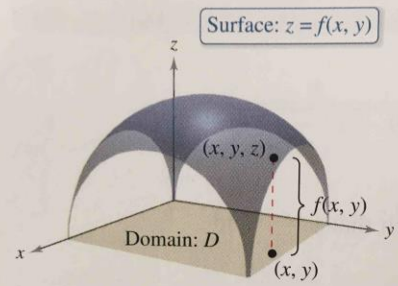


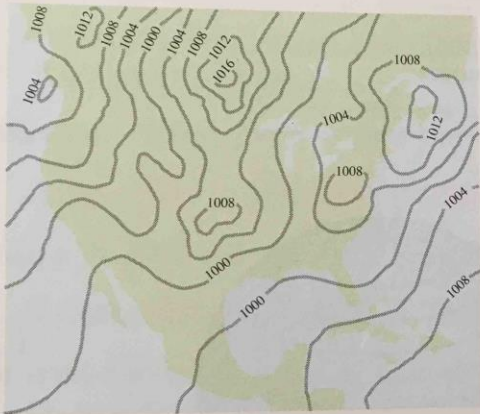
Figure 13.2

EXAMPLE 2 Describing the Graph of a Function of Two Variables

What is the range of

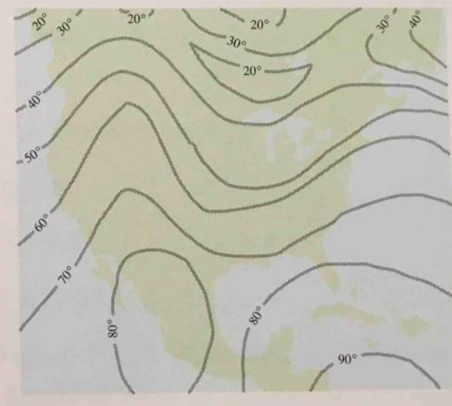
$$f(x, y) = \sqrt{16 - 4x^2 - y^2}?$$

Describe the graph of f.



Level curves show the lines of equal pressure (isobars), measured in millibars.

Figure 13.5



Level curves show the lines of equal temperature (isotherms), measured in degrees Fahrenheit.

Figure 13.6

Figure 13.7

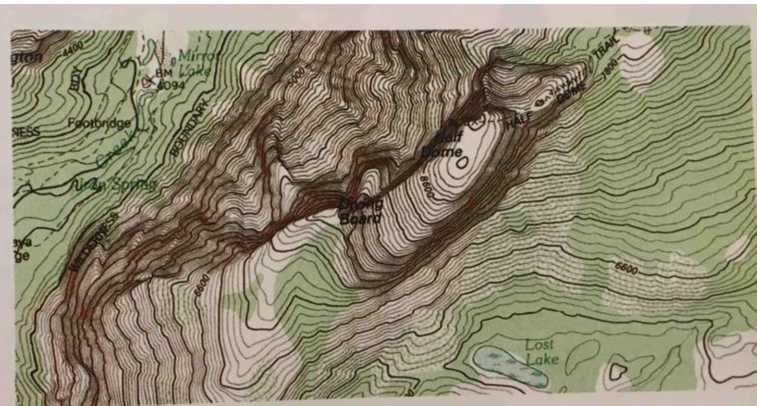


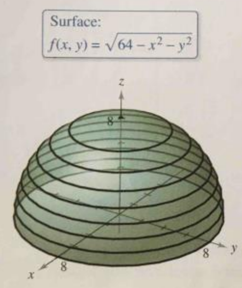
Figure 13.8

EXAMPLE 3 Sketching a Contour Map

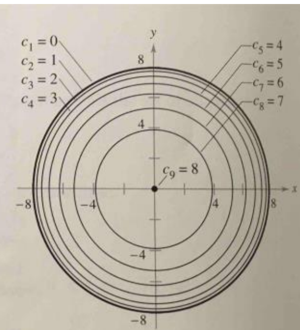
The hemisphere

$$f(x, y) = \sqrt{64 - x^2 - y^2}$$

is shown in Figure 13.9. Sketch a contour map of this surface using lev
corresponding to $c = 0, 1, 2, \ldots, 8$.



Hemisphere
Figure 13.9



Contour map
Figure 13.10

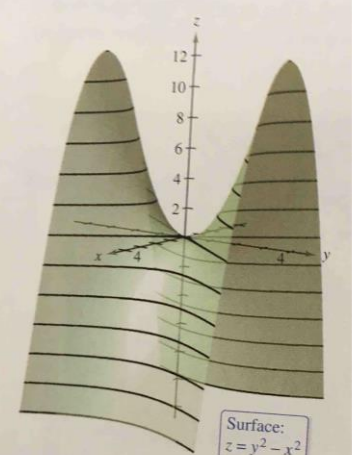
EXAMPLE 4 Sketching a Contour Map

⋮···▷ *See LarsonCalculus.com for an interactive version of this type of example.*

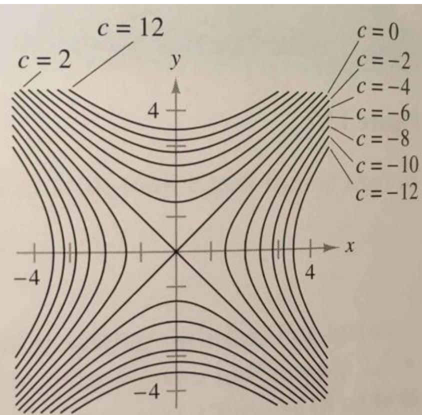
The hyperbolic paraboloid

$$z = y^2 - x^2$$

is shown in Figure 13.11. Sketch a contour map of this surface.



Hyperbolic paraboloid
Figure 13.11



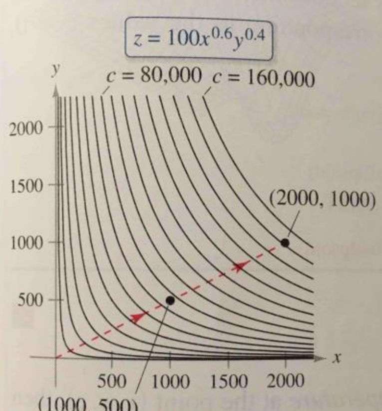
Hyperbolic level curves (at increments of 2)
Figure 13.12

EXAMPLE 5 **The Cobb-Douglas Production Function**

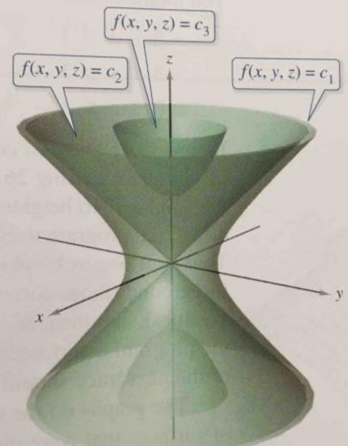
A toy manufacturer estimates a production function to be

$$f(x, y) = 100x^{0.6}y^{0.4}$$

where x is the number of units of labor and y is the number of units of capital. Compare the production level when $x = 1000$ and $y = 500$ with the production level when
00) $x = 2000$ and $y = 1000$.



Level curves (at increments of 10,000)
Figure 13.13



Level surfaces of f

Figure 13.14

EXAMPLE 6 Level Surfaces

Describe the level surfaces of

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

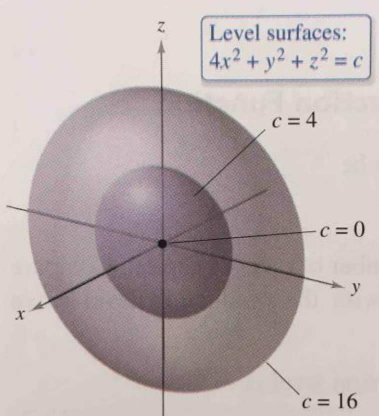
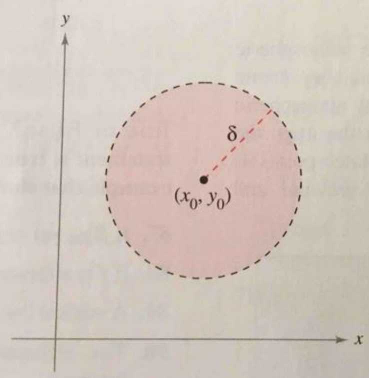


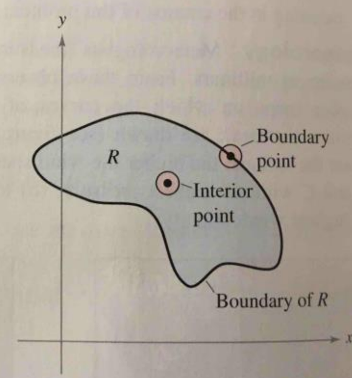
Figure 13.15

13.2 Limit Process:



An open disk

Figure 13.17



The boundary and interior points of a region R

Figure 13.18

Limit of a Function of Two Variables

Definition of the Limit of a Function of Two Variables

Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

13.3

Definition of Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$ Partial derivative with respect to x

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$ Partial derivative with respect to y

provided the limits exist.

EXAMPLE 1 Finding Partial Derivatives

Notation for First Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$ Partial derivative with respect to x

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$ Partial derivative with respect to y

The first partials evaluated at the point (a, b) are denoted by

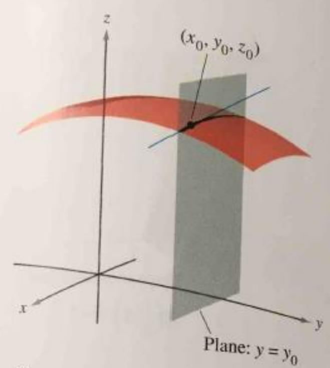
$$\left.\frac{\partial z}{\partial x}\right|_{(a, b)} = f_x(a, b)$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{(a, b)} = f_y(a, b).$$

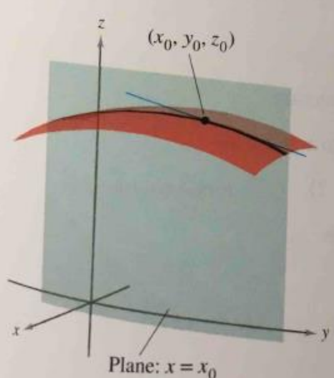
EXAMPLE 2 **Finding and Evaluating Partial Derivatives**

For $f(x, y) = xe^{x^2y}$, find f_x and f_y, and evaluate each at the point $(1, \ln 2)$.



$\frac{\partial f}{\partial x}$ = slope in x-direction

Figure 13.28



$\frac{\partial f}{\partial y}$ = slope in y-direction

Figure 13.29

EXAMPLE 3 **Finding the Slopes of a Surface**

▷ *See LarsonCalculus.com for an interactive version of this type of example.*

Find the slopes in the x-direction and in the y-direction of the surface

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.

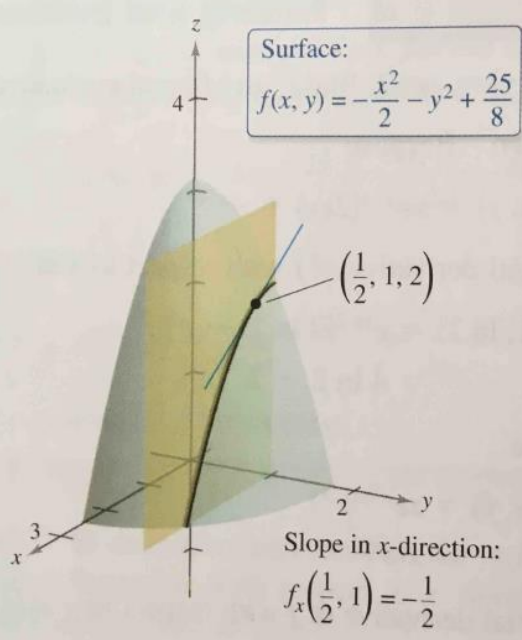


Figure 13.30

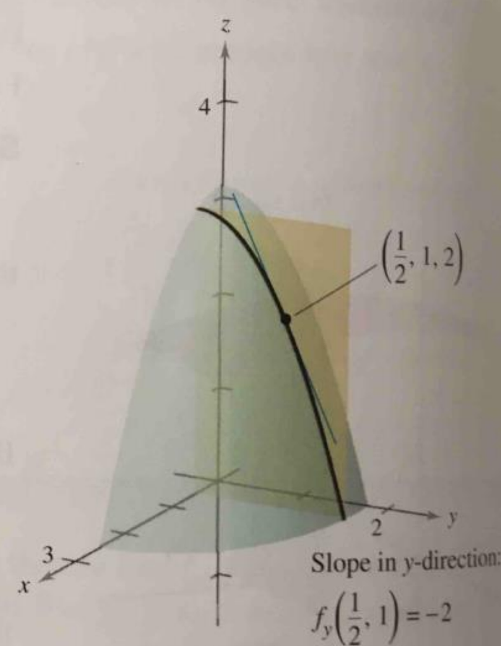


Figure 13.31

EXAMPLE 4 **Finding the Slopes of a Surface**

Find the slopes of the surface

$$f(x, y) = 1 - (x - 1)^2 - (y - 2)^2$$

at the point $(1, 2, 1)$ in the x-direction and in the y-direction.

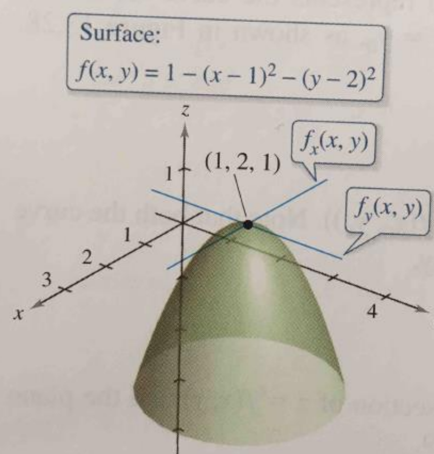
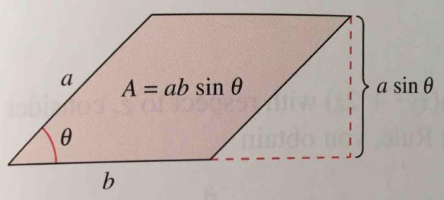


Figure 13.32



The area of the parallelogram is $ab \sin \theta$.

Figure 13.33

EXAMPLE 5 **Using Partial Derivatives to Find Rates of Change**

The area of a parallelogram with adjacent sides a and b and included angle θ is given by $A = ab \sin \theta$, as shown in Figure 13.33.

a. Find the rate of change of A with respect to a for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.

b. Find the rate of change of A with respect to θ for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.

EXAMPLE 6 Finding Partial Derivatives

a) $f(x, y, z) = xy + yz^2 + xz$

b) $f(x, y, z) = z \sin(xy^2 + 2z)$

c) $f(x, y, z, w) = \dfrac{x + y + z}{w}$

Higher-Order Partial Derivatives

As is true for ordinary derivatives, it is possible to take second, third, and highe[r] partial derivatives of a function of several variables, provided such derivatives Higher-order derivatives are denoted by the order in which the differentiation [...] For instance, the function $z = f(x, y)$ has the following second partial derivative[s]

1. Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

2. Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

3. Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

4. Differentiate first with respect to y and then with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives.**

EXAMPLE 7 Finding Second Partial Derivatives

Find the second partial derivatives of

$$f(x, y) = 3xy^2 - 2y + 5x^2y^2$$

and determine the value of $f_{xy}(-1, 2)$.

THEOREM 13.3 Equality of Mixed Partial Derivatives

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

EXAMPLE 8 Finding Higher-Order Partial Derivatives

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function

$$f(x, y, z) = ye^x + x \ln z.$$

A company manufactures two types of wood-burning stoves: a freestanding model and a fireplace-insert model. The cost function for producing x freestanding and y fireplace-insert stoves is

$$C = 32\sqrt{xy} + 175x + 205y + 1050.$$

(a) Find the marginal costs ($\partial C/\partial x$ and $\partial C/\partial y$) when $x = 80$ and $y = 20$.

(b) When additional production is required, which model of stove results in the cost increasing at a higher rate? How can this be determined from the cost model?