THEOREM 8.3 The Extended Mean Value Theorem

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 8.4 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 1 Indeterminate Form 0/0

Evaluate $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{x}$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

EXAMPLE 2 **Indeterminate Form ∞/∞**

Evaluate $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$.

EXAMPLE 3 **Applying L'Hôpital's Rule More than Once**

Evaluate $\displaystyle\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$.

EXAMPLE 4 **Indeterminate Form $0 \cdot \infty$**

Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$.

EXAMPLE 5 **Indeterminate Form 1^{∞}**

Evaluate $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$.

EXAMPLE 6 **Indeterminate Form 0^0**

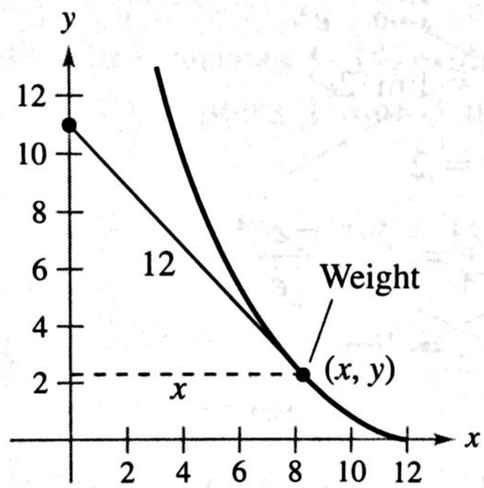
▷ *See LarsonCalculus.com for an interactive version of this type of example.*

Evaluate $\lim_{x \to 0^+} (\sin x)^x$.

EXAMPLE 7 **Indeterminate Form $\infty - \infty$**

Evaluate $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x - 1} \right)$.

90. Tractrix A person moves from the origin along the positive y-axis pulling a weight at the end of a 12-meter rope (see figure). Initially, the weight is located at the point $(12, 0)$.



(a) Show that the slope of the tangent line of the path of the weight is

$$\frac{dy}{dx} = -\frac{\sqrt{144 - x^2}}{x}.$$

(b) Use the result of part (a) to find the equation of the path of the weight. Use a graphing utility to graph the path and compare it with the figure.

(c) Find any vertical asymptotes of the graph in part (b).

(d) When the person has reached the point $(0, 12)$, how far has the weight moved?