4.2 Homogeneous Linear Equations: The General Solution

We begin our study of the linear second-order constant-coefficient differential equation

(1) $$ay'' + by' + cy = f(t) \quad (a \neq 0)$$

with the special case where the function $f(t)$ is zero:

(2) $$ay'' + by' + cy = 0 .$$

Example 1 Find a pair of solutions to

(4) $$y'' + 5y' - 6y = 0 .$$

Example 2 Solve the initial value problem

(6) $$y'' + 2y' - y = 0 ; \quad y(0) = 0 , \quad y'(0) = -1 .$$

Existence and Uniqueness: Homogeneous Case

Theorem 1. For any real numbers $a\ (\neq 0)$, b, c, t_0, Y_0, and Y_1, there exists a unique solution to the initial value problem

$$\textbf{(10)}\quad ay'' + by' + cy = 0\,; \qquad y(t_0) = Y_0\,, \qquad y'(t_0) = Y_1\,.$$

The solution is valid for all t in $(-\infty, +\infty)$.

Linear Independence of Two Functions

Definition 1. A pair of functions $y_1(t)$ and $y_2(t)$ is said to be **linearly independent on the interval I** if and only if neither of them is a constant multiple of the other on all of I.[††] We say that y_1 and y_2 are **linearly dependent on I** if one of them *is* a constant multiple of the other on all of I.

Representation of Solutions to Initial Value Problem

Theorem 2. If $y_1(t)$ and $y_2(t)$ are any two solutions to the differential equation (2) that are linearly independent on $(-\infty, \infty)$, then unique constants c_1 and c_2 can always be found so that $c_1y_1(t) + c_2y_2(t)$ satisfies the initial value problem (10) on $(-\infty, \infty)$.

The proof of Theorem 2 will be easy once we establish the following technical lemma.

A Condition for Linear Dependence of Solutions

Lemma 1. For any real numbers $a\ (\neq 0)$, b, and c, if $y_1(t)$ and $y_2(t)$ are any two solutions to the differential equation (2) on $(-\infty, \infty)$ and if the equality

$$\textbf{(11)}\qquad y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau) = 0$$

holds at *any* point τ, then y_1 and y_2 are linearly dependent on $(-\infty, \infty)$. (The expression on the left-hand side of (11) is called the *Wronskian* of y_1 and y_2 at the point τ; see Problem 34 on page 164.)

Distinct Real Roots

If the auxiliary equation (3) has distinct real roots r_1 and r_2, then both $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are solutions to (2) and $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is a general solution.

† To solve for c_1, for example, multiply the first equation by $y_2'(t_0)$ and the second by $y_2(t_0)$ and subtract.

Repeated Root

If the auxiliary equation (3) has a repeated root r, then both $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ are solutions to (2), and $y(t) = c_1 e^{rt} + c_2 t e^{rt}$ is a general solution.

We illustrate this result before giving its proof.

Example 3 Find a solution to the initial value problem

(12) $$y'' + 4y' + 4y = 0\,; \qquad y(0) = 1\,, \qquad y'(0) = 3\,.$$

Example 4 Find a general solution to

(14) $$y''' + 3y'' - y' - 3y = 0\,.$$

tions to

Example 1 Use the general solution (8) to solve the initial value problem

$$y'' + 2y' + 2y = 0\,; \qquad y(0) = 0, \qquad y'(0) = 2\ .$$

Maclaurin Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad for\,|x| < 1$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad for\,|x| < 1$$