THEOREM 12.6 Arc Length of a Space Curve

If C is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ on an interval $[a, b]$, then the arc length of C on the interval is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$

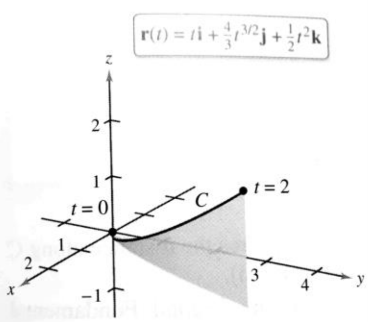
EXAMPLE 1 Finding the Arc Length of a Curve in Space

See LarsonCalculus.com for an interactive version of this type of example.

Find the arc length of the curve given by

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{3}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$

from $t = 0$ to $t = 2$, as shown in Figure 12.28.



As t increases from 0 to 2, the vector $\mathbf{r}(t)$ traces out a curve.

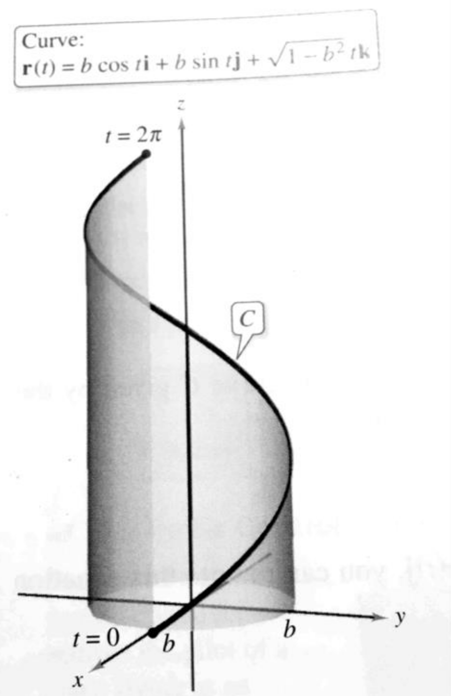
Figure 12.28

EXAMPLE 2 Finding the Arc Length of a Helix

Find the length of one turn of the helix given by

$$\mathbf{r}(t) = b\cos t\mathbf{i} + b\sin t\mathbf{j} + \sqrt{1-b^2}\,t\mathbf{k}$$

as shown in Figure 12.29.



One turn of a helix
Figure 12.29

Definition of Arc Length Function

Let C be a smooth curve given by $\mathbf{r}(t)$ defined on the closed interval $[a, b]$. For $a \leq t \leq b$, the **arc length function** is

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du.$$

The arc length s is called the **arc length parameter.** (See Figure 12.30.)

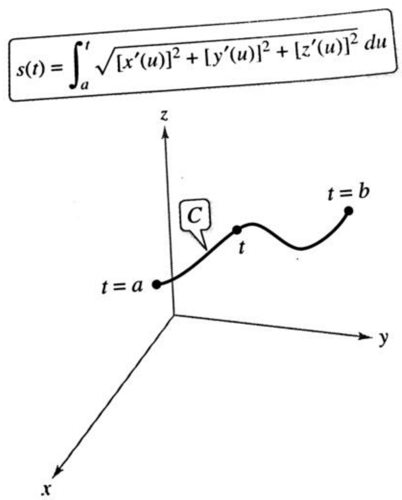


Figure 12.30

efinition of Arc Length Function

t C be a smooth curve given by $\mathbf{r}(t)$ defined on the closed interval $[a, b$
$\leq t \leq b$, the **arc length function** is

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du.$$

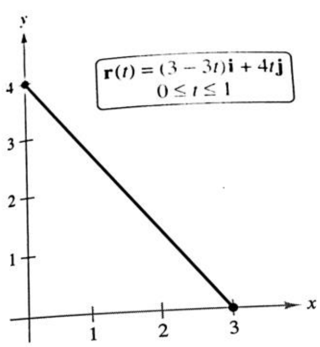
e arc length s is called the **arc length parameter.** (See Figure 12.30.

Finding the Arc Length Function for a Line

Find the arc length function $s(t)$ for the line segment given by

$$\mathbf{r}(t) = (3 - 3t)\mathbf{i} + 4t\mathbf{j}, \quad 0 \leq t \leq 1$$

and write $\mathbf{r}$ as a function of the parameter s. (See Figure 12.31.)



The line segment from (3, 0) to (0, 4) can be parametrized using the arc length parameter s.
Figure 12.31

THEOREM 12.7 Arc Length Parameter

If C is a smooth curve given by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} \qquad \text{Plane curve}$$

or

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k} \qquad \text{Space curve}$$

where s is the arc length parameter, then

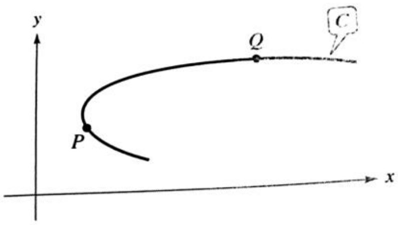
$$\|\mathbf{r}'(s)\| = 1.$$

→ Moreover, if t is *any* parameter for the vector-valued function $\mathbf{r}$ such that $\|\mathbf{r}'(t)\| = 1$, then t must be the arc length parameter.

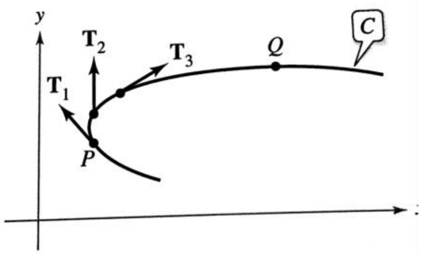
Definition of Curvature

Let C be a smooth curve (in the plane *or* in space) given by $\mathbf{r}(s)$, where s is the arc length parameter. The **curvature** K at s is

$$K = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|.$$



Curvature at P is greater than at Q.
Figure 12.32

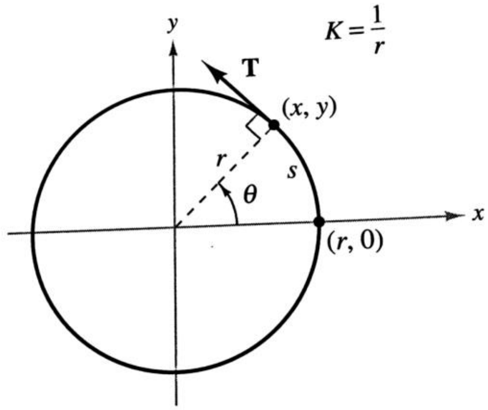


The magnitude of the rate of change o
T with respect to the arc length is the
curvature of a curve.
Figure 12.33

EXAMPLE 4 Finding the Curvature of a Circle

Show that the curvature of a circle of radius r is

$$K = \frac{1}{r}.$$

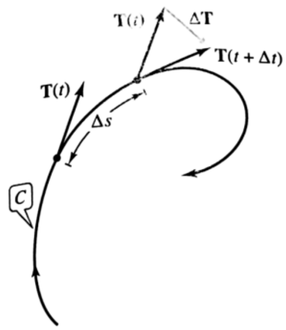


The curvature of a circle is constant.
Figure 12.34

THEOREM 12.8 Formulas for Curvature

If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature K of C at t is

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$



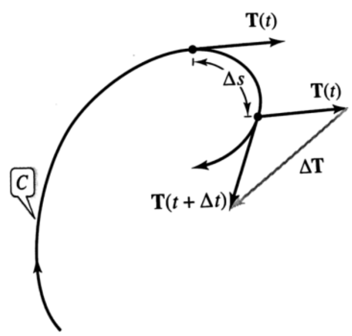


Figure 12.35

EXAMPLE 5 **Finding the Curvature of a Space Curve**

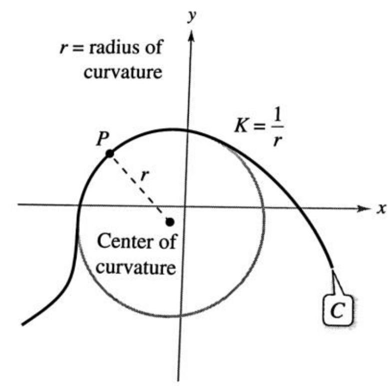
Find the curvature of the curve given by

$$\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \frac{1}{3}t^3\mathbf{k}.$$

THEOREM 12.9 Curvature in Rectangular Coordinates

If C is the graph of a twice-differentiable function given by $y = f(x)$, then the curvature K at the point (x, y) is

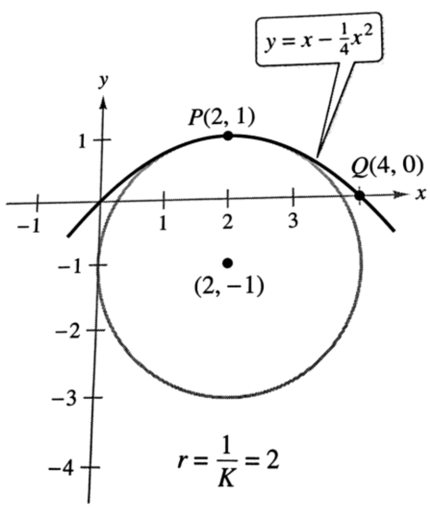
$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$



The circle of curvature
Figure 12.36

EXAMPLE 6 Finding Curvature in Rectangular Coordinates

Find the curvature of the parabola given by $y = x - \frac{1}{4}x^2$ at $x = 2$. Sketch the circle of curvature at $(2, 1)$.



The circle of curvature
Figure 12.37

THEOREM 12.10 Acceleration, Speed, and Curvature

If $\mathbf{r}(t)$ is the position vector for a smooth curve C, then the acceleration vector is given by

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T} + K\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

where K is the curvature of C and ds/dt is the speed.

EXAMPLE 7 Tangential and Normal Components of Acceleration

Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for the curve given by

$$\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \tfrac{1}{3}t^3\mathbf{k}.$$

EXAMPLE 8 Frictional Force

A 360-kilogram go-cart is driven at a speed of 60 kilometers per hour around a circular racetrack of radius 12 meters, as shown in Figure 12.40. To keep the cart from skidding off course, what frictional force must the track surface exert on the tires?