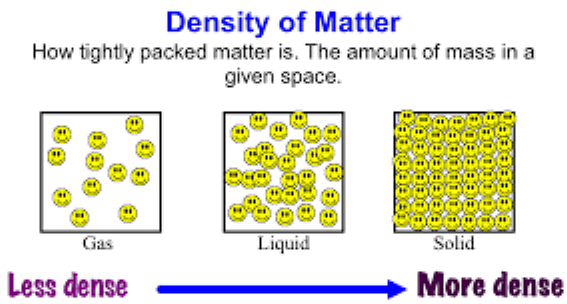
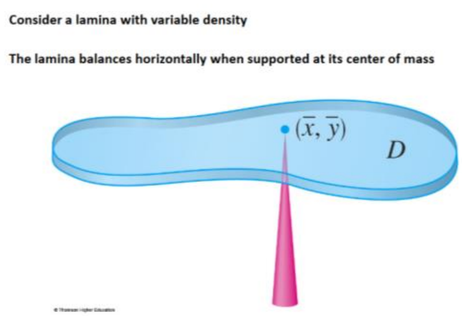




Center of Mass in Two Dimensions

The center of mass is also known as the center of gravity if the object is in a uniform gravitational field. If the object has uniform density, the center of mass is the geometric center of the object, which is called the centroid. Figure shows a point P as the center of mass of a lamina. The lamina is perfectly balanced about its center of mass.

$$\text{Mass} = \rho A = \rho \int_R \int dA = \int_R \int \rho \, dA. \qquad \text{Constant density}$$

Definition of Mass of a Planar Lamina of Variable Density

If ρ is a continuous density function on the lamina corresponding to a plane region R, then the mass m of the lamina is given by

$$m = \int_R \int \rho(x, y) \, dA. \qquad \text{Variable density}$$

EXAMPLE 1 **Finding the Mass of a Planar Lamina**

Find the mass of the triangular lamina with vertices $(0, 0)$, $(0, 3)$, and $(2, 3)$, given that the density at (x, y) is $\rho(x, y) = 2x + y$.

Moments and Center of Mass of a Variable Density Planar Lamina

Let ρ be a continuous density function on the planar lamina R. The **moments of mass** with respect to the x- and y-axes are

$$M_x = \int_R\int y\rho(x, y)\, dA$$

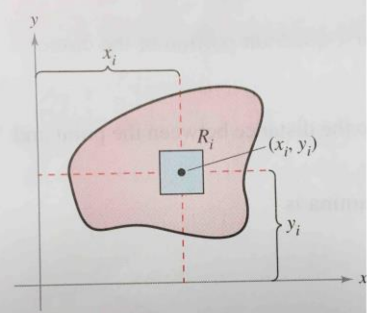
and

$$M_y = \int_R\int x\rho(x, y)\, dA.$$

If m is the mass of the lamina, then the **center of mass** is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

If R represents a simple plane region rather than a lamina, then the point $(\bar{x}, \bar{y})$ is called the **centroid** of the region.



$M_x = (\text{mass})(y_i)$
$M_y = (\text{mass})(x_i)$

Figure 14.36

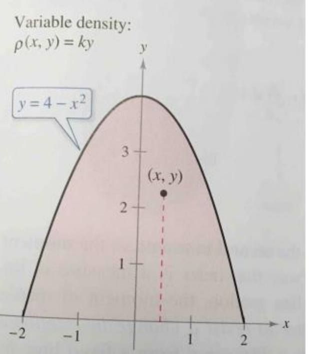
EXAMPLE 3 Finding the Center of Mass

See LarsonCalculus.com for an interactive version of this type of example.

Find the center of mass of the lamina corresponding to the parabolic region

$$0 \le y \le 4 - x^2 \qquad \text{Parabolic region}$$

where the density at the point (x, y) is proportional to the distance between (x, y) and the x-axis, as shown in Figure 14.38.



Parabolic region of variable density
Figure 14.38

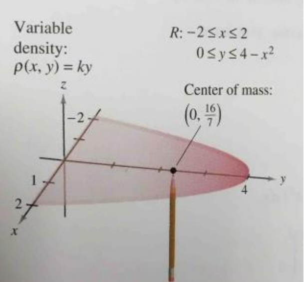


Figure 14.39

Definition of Triple Integral

If f is continuous over a bounded solid region Q, then the **triple integral of f over Q** is defined as

$$\iiint_Q f(x, y, z)\, dV = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta V_i$$

provided the limit exists. The **volume** of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q dV.$$

y

THEOREM 14.4 Evaluation by Iterated Integrals

Let f be continuous on a solid region Q defined by

$$a \le x \le b,$$
$$h_1(x) \le y \le h_2(x),$$
$$g_1(x, y) \le z \le g_2(x, y)$$

where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint_Q f(x, y, z)\, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z)\, dz\, dy\, dx.$$

EXAMPLE 1 **Evaluating a Triple Iterated Integral**

Evaluate the triple iterated integral

$$\int_0^2 \int_0^x \int_0^{x+y} e^x (y + 2z)\, dz\, dy\, dx.$$

EXAMPLE 3 **Changing the Order of Integration**

Evaluate $\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\, dz\, dy\, dx.$

Q:

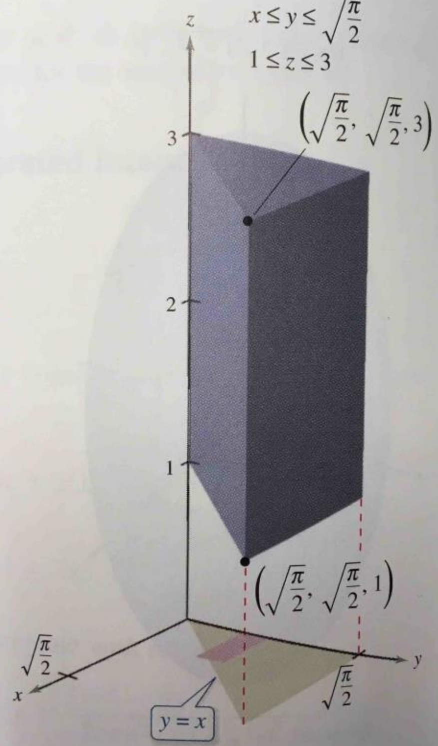


Figure 14.56

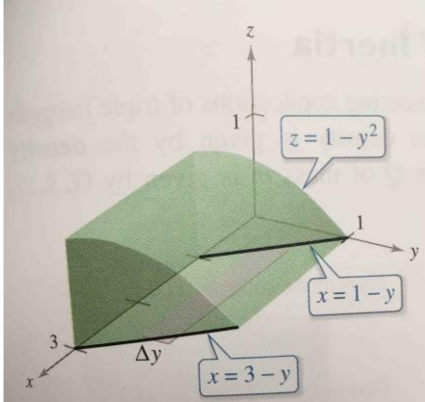
EXAMPLE 4 **Determining the Limits of Integration**

Set up a triple integral for the volume of each solid region.

a. The region in the first octant bounded above by the cylinder $z = 1 - y^2$ and lying between the vertical planes $x + y = 1$ and $x + y = 3$

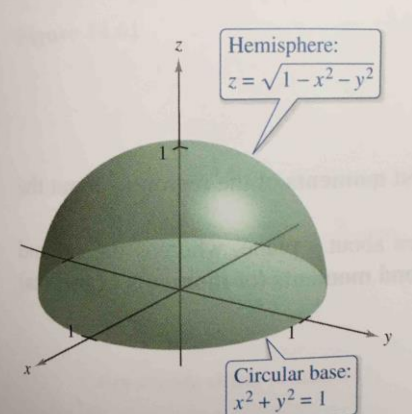
b. The upper hemisphere $z = \sqrt{1 - x^2 - y^2}$

c. The region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$



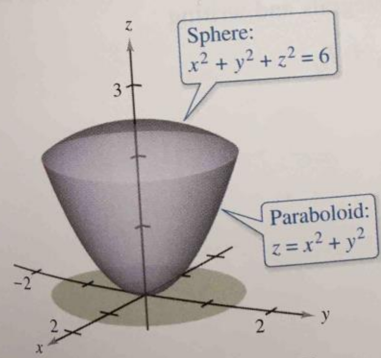
$$Q:\ 0 \le z \le 1 - y^2$$
$$1 - y \le x \le 3 - y$$
$$0 \le y \le 1$$

Figure 14.57



$$Q:\ 0 \le z \le \sqrt{1 - x^2 - y^2}$$
$$-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2}$$
$$-1 \le y \le 1$$

Figure 14.58



Q: $x^2 + y^2 \le z \le \sqrt{6 - x^2 - y^2}$
$-\sqrt{2 - x^2} \le y \le \sqrt{2 - x^2}$
$-\sqrt{2} \le x \le \sqrt{2}$

Figure 14.59