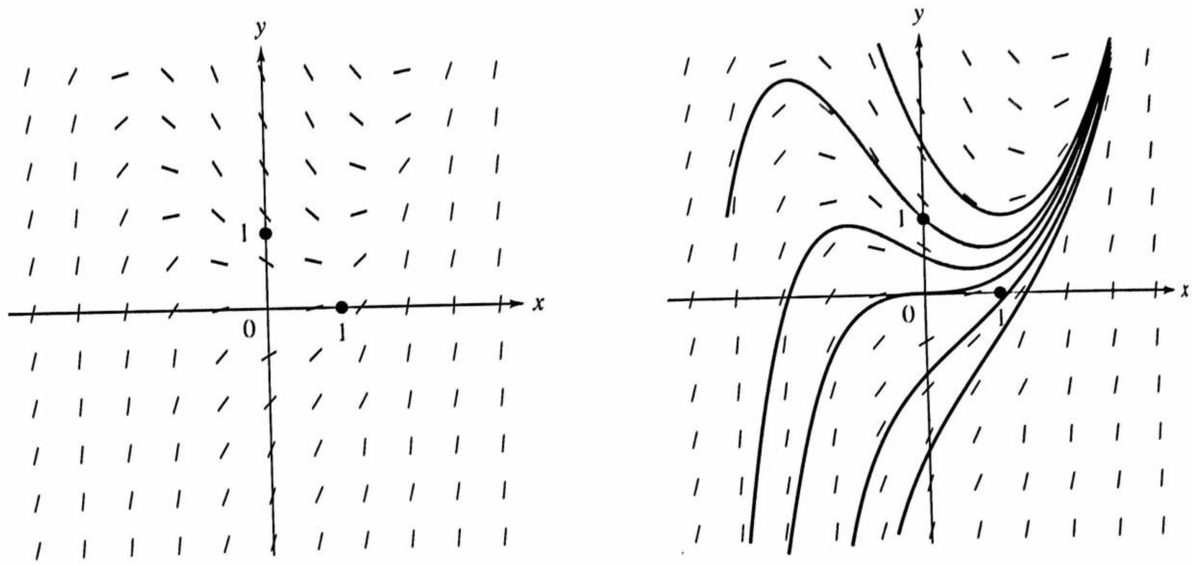
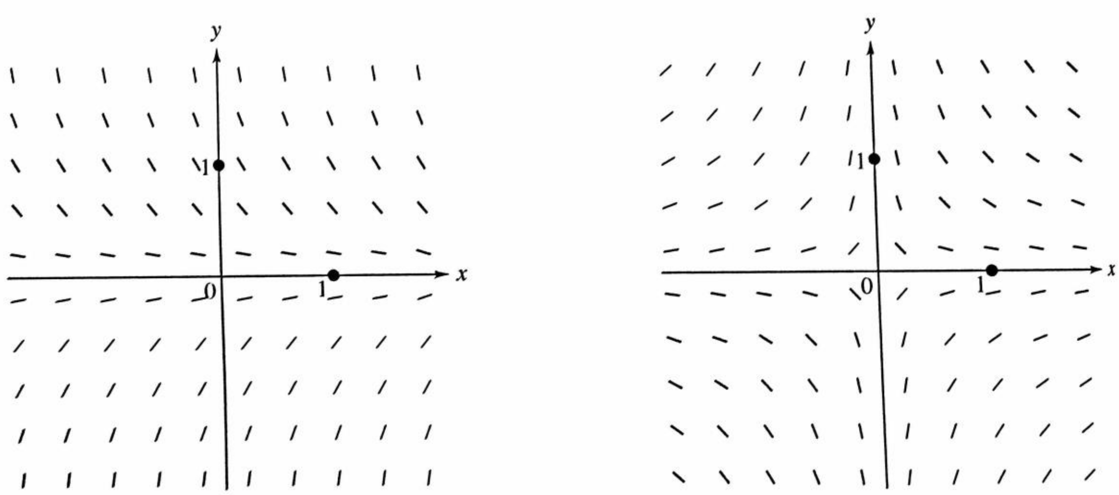
$$\textbf{(1)} \qquad \frac{dy}{dx} = x^2 - y\,.$$



(a) (b)

Figure 1.6 (a) Direction field for $dy/dx = x^2 - y$ (b) Solutions to $dy/dx = x^2 - y$



(a) $\frac{dy}{dx} = -2y$ (b) $\frac{dy}{dx} = -\frac{y}{x}$

Figure 1.7 (a) Direction field for $dy/dx = -2y$ (b) Direction field for $dy/dx = -y/x$

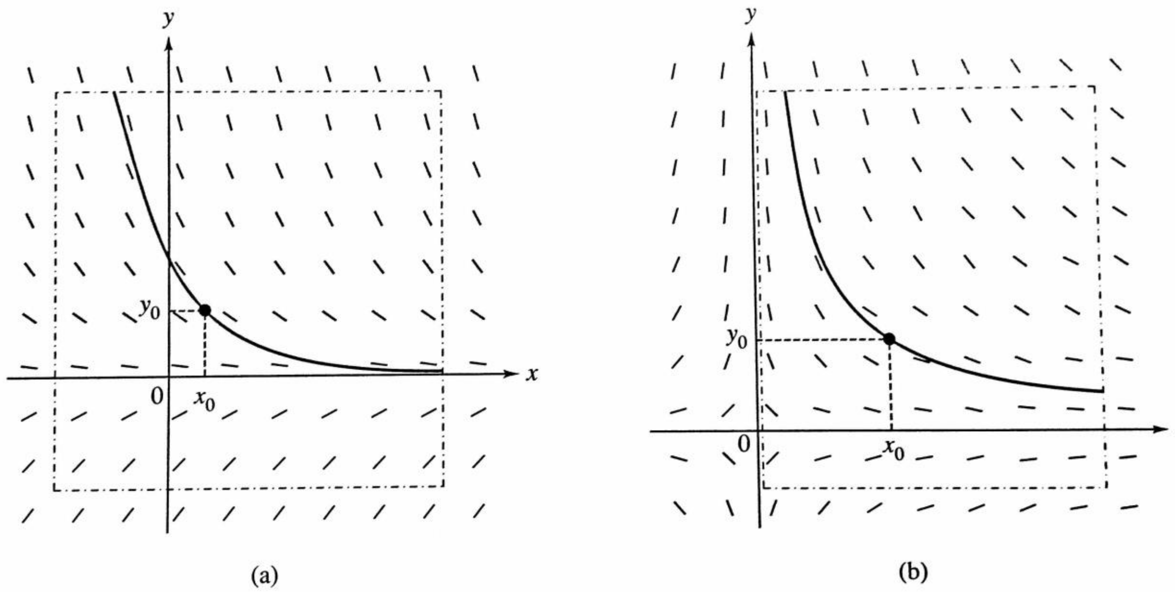


Figure 1.8 (a) A solution for $dy/dx = -2y$ (b) A solution for $dy/dx = -y/x$

Existence and Uniqueness of Solution

Theorem 1. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \ , \qquad y(x_0) = y_0 \ .$$

If f and $\partial f/\partial y$ are continuous functions in some rectangle

$$R = \{(x, y) \colon a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.†

Example 1 The *logistic equation* for the population p (in thousands) at time t of a certain species is given by

$$\frac{dp}{dt} = p(2-p)\ . \tag{2}$$

(Of course, p is nonnegative. The interpretation of the terms in the logistic equation is discussed in Section 3.2.) From the direction field sketched in Figure 1.10 on page 19, answer the following:

(a) If the initial population is 3000 [that is, $p(0) = 3$], what can you say about the limiting population $\lim_{t\to+\infty} p(t)$?

(b) Can a population of 1000 ever decline to 500?

(c) Can a population of 1000 ever increase to 3000?

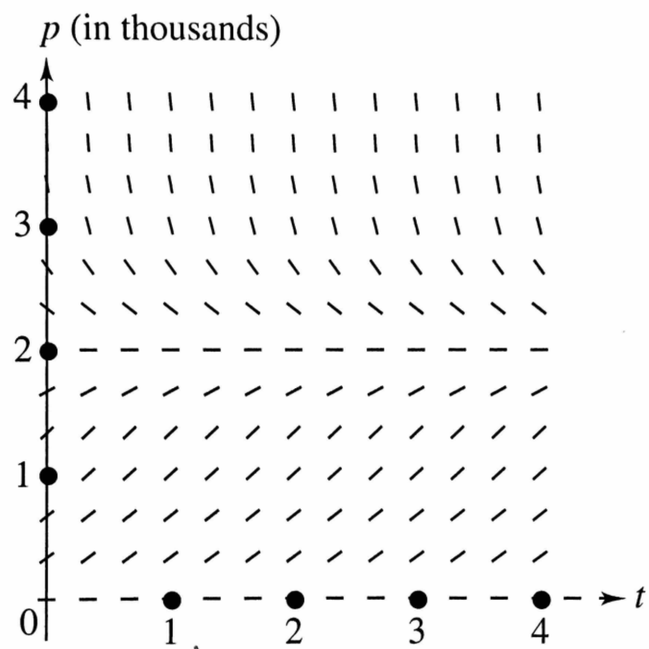


Figure 1.10 Direction field for logistic equation

Separable Equation

Definition 1. If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y, then the differential equation is called **separable.**†

Method for Solving Separable Equations

To solve the equation

$$\frac{dy}{dx} = g(x)p(y) \tag{2}$$

multiply by dx and by $h(y) := 1/p(y)$ to obtain

$$h(y)\,dy = g(x)\,dx\,.$$

Then integrate both sides:

$$\int h(y)\,dy = \int g(x)\,dx\,,$$

$$H(y) = G(x) + C\,, \tag{3}$$

where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Linear Equations

A type of first-order differential equation that occurs frequent
equation. Recall from Section 1.1 that a **linear first-order equa**
expressed in the form

$$\textbf{(1)} \qquad a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)\,,$$

$$\textbf{(2)} \qquad a_1(x)\frac{dy}{dx} = b(x)\,,$$

which is equivalent to

$$y(x) = \int \frac{b(x)}{a_1(x)}dx + C$$

$$\textbf{(3)} \qquad \frac{d}{dx}[a_1(x)y] = b(x)$$

and the solution is again elementary:

$$a_1(x)y = \int b(x)\,dx + C\,,$$

$$y(x) = \frac{1}{a_1(x)}\left[\int b(x)\,dx + C\right].$$

Method for Solving Linear Equations

(a) Write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x) .$$

(b) Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)dx\right] .$$

(c) Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{d}{dx}[\mu(x)y]$, obtain

$$\underbrace{\mu(x)\frac{dy}{dx} + P(x)\mu(x)y}_{} = \mu(x)Q(x) ,$$

$$\frac{d}{dx}[\mu(x)y] \qquad = \mu(x)Q(x) .$$

(d) Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain (8).

Example 1 Find the general solution to

$$\textbf{(9)} \qquad \frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x , \qquad x > 0 .$$

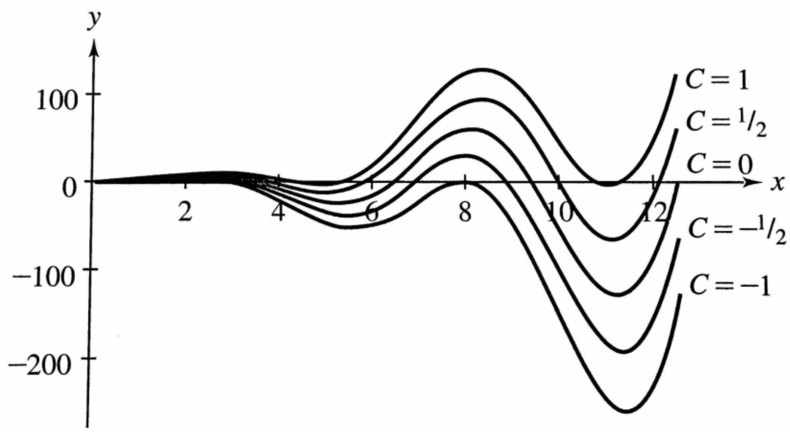


Figure 2.5 Graph of $y = x^2 \sin x + Cx^2$ for five values of the constant C

Example 2 A rock contains two radioactive isotopes, RA_1 and RA_2, that belong to the same radioactive series; that is, RA_1 decays into RA_2, which then decays into stable atoms. Assume that the rate at which RA_1 decays into RA_2 is $50e^{-10t}$ kg/sec. Because the rate of decay of RA_2 is proportional to the mass $y(t)$ of RA_2 present, the rate of change in RA_2 is

$$\frac{dy}{dt} = \text{rate of creation} - \text{rate of decay}\,,$$

$$\frac{dy}{dt} = 50e^{-10t} - ky\,, \tag{12}$$

where $k > 0$ is the decay constant. If $k = 2/\text{sec}$ and initially $y(0) = 40$ kg, find the mass $y(t)$ of RA_2 for $t \geq 0$.