Superposition Principle

Theorem 3. If y_1 is a solution to the differential equation

$$ay'' + by' + cy = f_1(t) ,$$

and y_2 is a solution to

$$ay'' + by' + cy = f_2(t) ,$$

then for any constants k_1 and k_2, the function $k_1y_1 + k_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t) .$$

Existence and Uniqueness: Nonhomogeneous Case

Theorem 4. For any real numbers $a(\neq 0)$, b, c, t_0, Y_0, and Y_1, suppose $y_p(t)$ is a particular solution to (3) in an interval I containing t_0 and that $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the associated homogeneous equation (4) in I. Then there exists a unique solution in I to the initial value problem

(6) $$ay'' + by' + cy = f(t), \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1 ,$$

and it is given by (5), for the appropriate choice of the constants c_1, c_2.

Example 2 Given that $y_p(t) = t^2$ is a particular solution to

$$y'' - y = 2 - t^2 ,$$

find a general solution and a solution satisfying $y(0) = 1, y'(0) = 0$.

Method of Undetermined Coefficients (Revisited)

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{rt},$$

where $P_m(t)$ is a polynomial of degree m, use the form

$$\textbf{(13)} \qquad y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt};$$

if r is not a root of the associated auxiliary equation, take $s = 0$; if r is a simple root of the associated auxiliary equation, take $s = 1$; and if r is a double root of the associated auxiliary equation, take $s = 2$.

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos\beta t + Q_n(t)e^{\alpha t}\sin\beta t, \quad \beta \neq 0,$$

where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n, use the form

$$\textbf{(14)} \qquad y_p(t) = t^s(A_k t^k + \cdots + A_1 t + A_0)e^{\alpha t}\cos\beta t + t^s(B_k t^k + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t,$$

where k is the larger of m and n. If $\alpha + i\beta$ is not a root of the associated auxiliary equation, take $s = 0$; if $\alpha + i\beta$ is a root of the associated auxiliary equation, take $s = 1$.

Example 5 Write down the form of a particular solution to the equation

$$y'' + 2y' + 2y = 5e^{-t}\sin t + 5t^3e^{-t}\cos t.$$

Example 6 Write down the form of a particular solution to the equation

$$y''' + 2y'' + y' = 5e^{-t}\sin t + 3 + 7te^{-t}.$$

Then we know that a general solution to this homogeneous equation is given by

(2) $$y_h(t) = c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are constants. To find a particular solution to the nonhomogeneous equation, the strategy of variation of parameters is to replace the constants in (2) by functions of t. That is, we seek a solution of (1) of the form†

(3) $$\mathbf{y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)}.$$

Because we have introduced two unknown functions, $v_1(t)$ and $v_2(t)$, it is reasonable to expect that we can impose two equations (requirements) on these functions. Naturally, one of these equations should come from (1). Let's therefore plug $y_p(t)$ given by (3) into (1). To accomplish this, we must first compute $y_p'(t)$ and $y_p''(t)$. From (3) we obtain

$$y_p' = (v_1' y_1 + v_2' y_2) + (v_1 y_1' + v_2 y_2').$$

To simplify the computation and to avoid second-order derivatives for the unknowns v_1, v_2 in the expression for y_p'', we impose the requirement

(4) $$v_1' y_1 + v_2' y_2 = 0.$$

Thus, the formula for y_p' becomes

(5) $$y_p' = v_1 y_1' + v_2 y_2',$$

and so

(6) $$y_p'' = v_1' y_1' + v_1 y''_1 + v_2' y_2' + v_2 y_2''.$$

Now, substituting y_p, y_p', and y_p'', as given in (3), (5), and (6), into (1), we find

(7)
$$\begin{aligned} f &= a y_p'' + b y_p' + c y_p \\ &= a(v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'') + b(v_1 y_1' + v_2 y_2') + c(v_1 y_1 + v_2 y_2) \\ &= a(v_1' y_1' + v_2' y_2') + v_1(a y_1'' + b y_1' + c y_1) + v_2(a y_2'' + b y_2' + c y_2) \\ &= a(v_1' y_1' + v_2' y_2') + 0 + 0 \end{aligned}$$

since y_1 and y_2 are solutions to the homogeneous equation. Thus, (7) reduces to

(8) $$v_1' y_1' + v_2' y_2' = \frac{f}{a}.$$

To summarize, if we can find v_1 and v_2 that satisfy both (4) and (8), that is,

(9)
$$\begin{aligned} \mathbf{y_1 v_1' + y_2 v_2'} &= \mathbf{0}, \\ \mathbf{y_1' v_1' + y_2' v_2'} &= \mathbf{\frac{f}{a}}. \end{aligned}$$

then y_p given by (3) will be a particular solution to (1). To determine v_1 and v_2, we first solve the linear system (9) for v_1' and v_2'. Algebraic manipulation or Cramer's rule (see Appendix D) immediately gives

$$v_1'(t) = \frac{-f(t) y_2(t)}{a[y_1(t) y_2'(t) - y_1'(t) y_2(t)]} \quad \text{and} \quad v_2'(t) = \frac{f(t) y_1(t)}{a[y_1(t) y_2'(t) - y_1'(t) y_2(t)]},$$

†In Exercises 2.3, Problem 36, we developed this approach for first-order li...

where the bracketed expression in the denominator (the Wronskian) is never zero because of Lemma 1, Section 4.2. Upon integrating these equations, we finally obtain

(10)
$$v_1(t) = \int \frac{-f(t)y_2(t)}{a\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]}\,dt \quad \text{and} \quad v_2(t) = \int \frac{f(t)y_1(t)}{a\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]}\,dt\,.$$

Let's review this procedure.

Method of Variation of Parameters

To determine a particular solution to $ay'' + by' + cy = f$:

(a) Find two linearly independent solutions $\{y_1(t), y_2(t)\}$ to the corresponding homogeneous equation and take

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)\,.$$

(b) Determine $v_1(t)$ and $v_2(t)$ by solving the system in (9) for $v_1'(t)$ and $v_2'(t)$ and integrating.

(c) Substitute $v_1(t)$ and $v_2(t)$ into the expression for $y_p(t)$ to obtain a particular solution.

Example 1 Find a general solution on $(-\pi/2, \pi/2)$ to

(11)
$$\frac{d^2y}{dt^2} + y = \tan t\,.$$

Example 2 Find a particular solution on $(-\pi/2, \pi/2)$ to

$$\textbf{(16)} \qquad \frac{d^2y}{dt^2} + y = \tan t + 3t - 1 \ .$$

Example 3 Find a particular solution of the variable coefficient linear equation

$$\textbf{(19)} \qquad t^2y'' - 4ty' + 6y = 4t^3 \ , \quad t > 0 \ ,$$

given that $y_1(t) = t^2$ and $y_2(t) = t^3$ are solutions to the corresponding homogeneous equation.