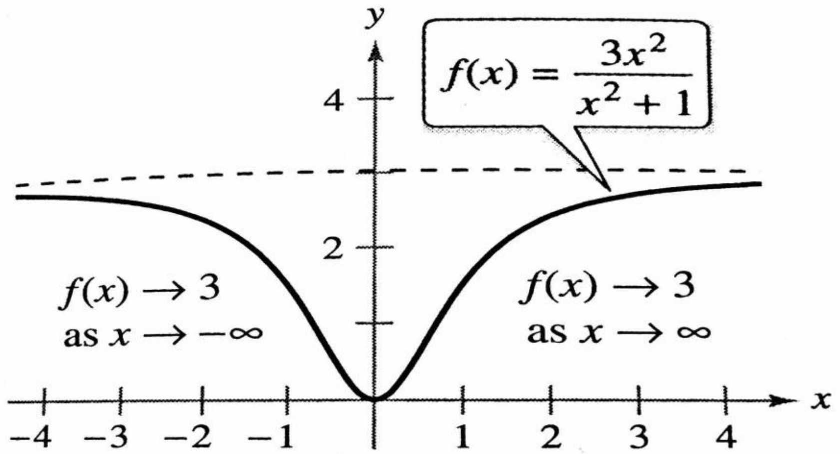


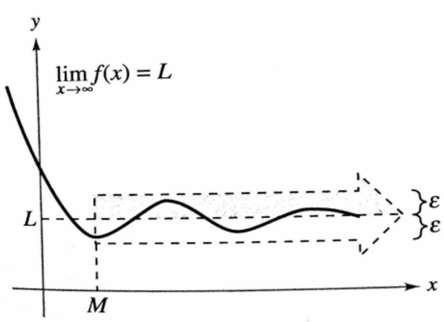
The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.

Figure 3.32

Definition of Limits at Infinity

Let L be a real number.

1. The statement $\lim_{x\to\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim_{x\to-\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.



$f(x)$ is within ε units of L as $x \to \infty$.

Figure 3.33

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L.$$

THEOREM 3.10 Limits at Infinity

If r is a positive rational number and c is any real number, then

$$\lim_{x \to \infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \to -\infty} \frac{c}{x^r} = 0.$$

A proof of this theorem is given in Appendix A.
See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 1 Finding a Limit at Infinity

Find the limit: $\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right)$.

EXAMPLE 2 Finding a Limit at Infinity

Find the limit: $\lim_{x \to \infty} \frac{2x - 1}{x + 1}$.

EXAMPLE 3 A Comparison of Three Rational Functions

See LarsonCalculus.com for an interactive version of this type of example.

Find each limit.

a. $\lim_{x\to\infty} \frac{2x + 5}{3x^2 + 1}$ **b.** $\lim_{x\to\infty} \frac{2x^2 + 5}{3x^2 + 1}$ **c.** $\lim_{x\to\infty} \frac{2x^3 + 5}{3x^2 + 1}$

GUIDELINES FOR FINDING LIMITS AT $\pm\infty$ OF RATIONAL FUNCTIONS

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

EXAMPLE 4 **A Function with Two Horizontal Asymptotes**

Find each limit.

a. $\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ **b.** $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

EXAMPLE 5 Limits Involving Trigonometric Functions

Find each limit.

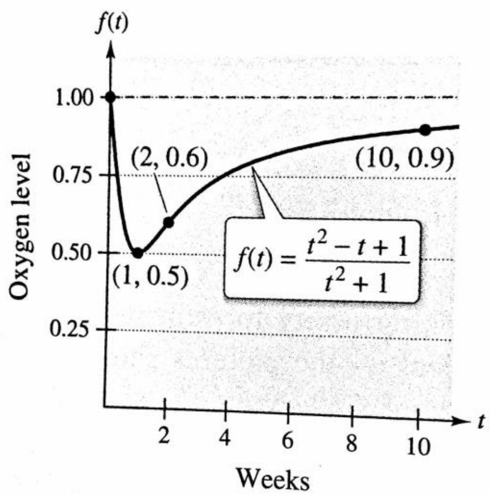
a. $\lim_{x\to\infty} \sin x$ **b.** $\lim_{x\to\infty} \frac{\sin x}{x}$

EXAMPLE 6 Oxygen Level in a Pond

Let $f(t)$ measure the level of oxygen in a pond, where $f(t) = 1$ is the normal (unpolluted) level and the time t is measured in weeks. When $t = 0$, organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is

$$f(t) = \frac{t^2 - t + 1}{t^2 + 1}.$$

What percent of the normal level of oxygen exists in the pond after 1 week? After 2 weeks? After 10 weeks? What is the limit as t approaches infinity?



The level of oxygen in a pond approaches the normal level of 1 as t approaches ∞.

Figure 3.41

Definition of Infinite Limits at Infinity

Let f be a function defined on the interval (a, ∞).

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements

$$\lim_{x\to-\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

EXAMPLE 7 **Finding Infinite Limits at Infinity**

Find each limit.

a. $\lim_{x\to\infty} x^3$ **b.** $\lim_{x\to-\infty} x^3$

EXAMPLE 8 **Finding Infinite Limits at Infinity**

Find each limit.

a. $\lim_{x\to\infty} \dfrac{2x^2 - 4x}{x + 1}$ **b.** $\lim_{x\to-\infty} \dfrac{2x^2 - 4x}{x + 1}$

Sketching a Graph **In Exercises 59–74, sketch the graph of the equation using extrema, intercepts, symmetry, and asymptotes. Then use a graphing utility to verify your result.**

59. $y = \dfrac{x}{1 - x}$

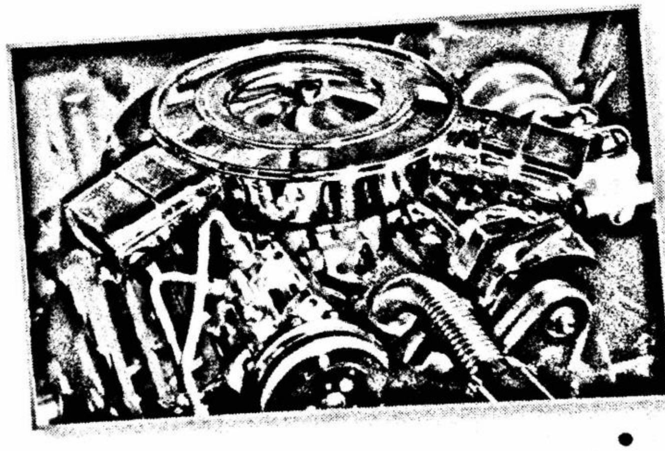
60. $y = \dfrac{x - 4}{x - 3}$

85. Engine Efficiency

The efficiency of an internal combustion engine is

$$\text{Efficiency } (\%) = 100\left[1 - \frac{1}{(v_1/v_2)^c}\right]$$

where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

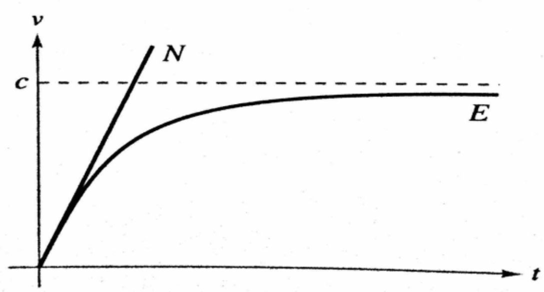


86. Average Cost A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

$$\overline{C} = \frac{C}{x}.$$

Find the limit of $\overline{C}$ as x approaches infinity.

87. Physics Newton's First Law of Motion and Einstein's Special Theory of Relativity differ concerning a particle's behavior as its velocity approaches the speed of light c. In the graph, functions N and E represent the velocity v, with respect to time t, of a particle accelerated by a constant force as predicted by Newton and Einstein, respectively. Write limit statements that describe these two theories.



x