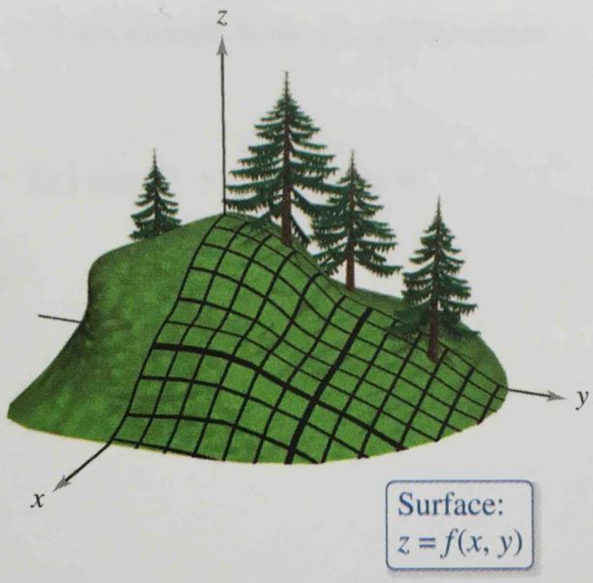


Figure 13.42

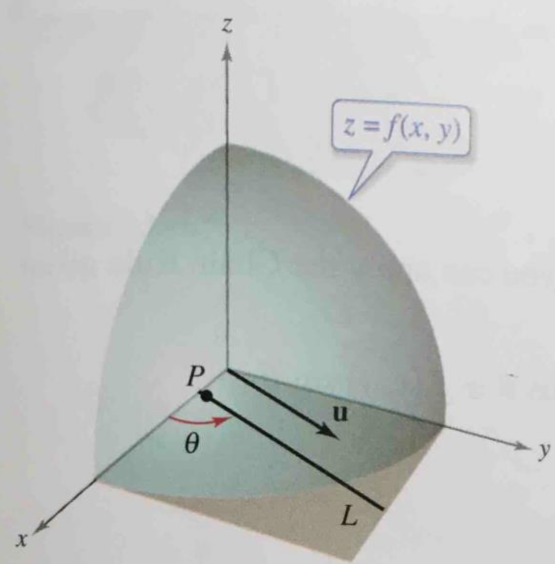


Figure 13.43

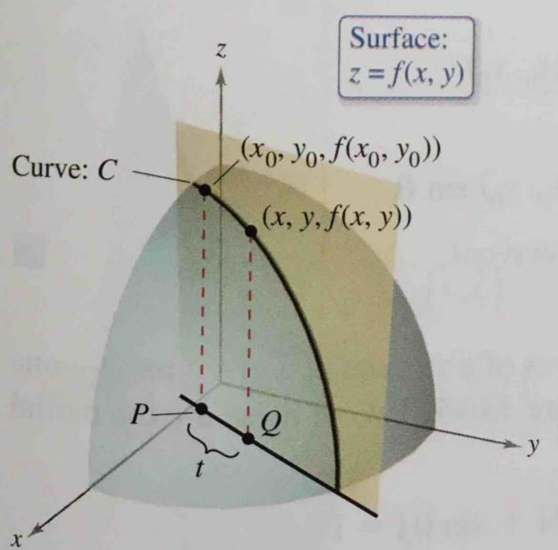


Figure 13.44

Definition of Directional Derivative

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of u,** denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t\to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

provided this limit exists.

THEOREM 13.9 Directional Derivative

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

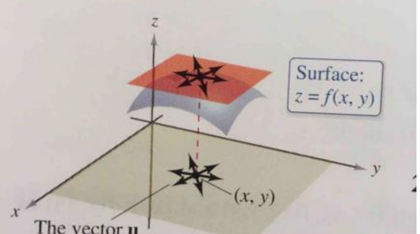


Figure 13.45

EXAMPLE 1 Finding a Directional Derivative

Find the directional derivative of

$$f(x, y) = 4 - x^2 - \frac{1}{4}y^2 \qquad \text{Surface}$$

at (1, 2) in the direction of

$$\mathbf{u} = \left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j}. \qquad \text{Direction}$$

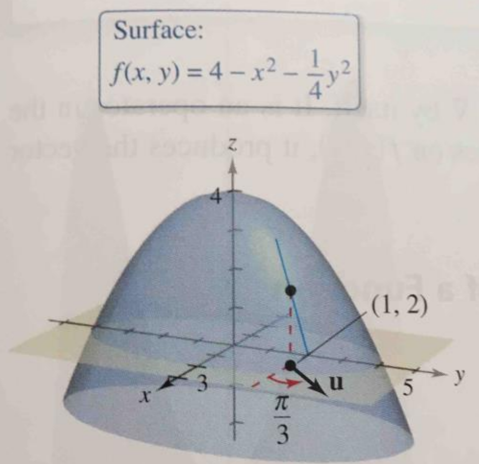


Figure 13.46

EXAMPLE 2 **Finding a Directional Derivative**

See LarsonCalculus.com for an interactive version of this type of example.

Find the directional derivative of

$$f(x, y) = x^2 \sin 2y \qquad \text{Surface}$$

at $(1, \pi/2)$ in the direction of

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}. \qquad \text{Direction}$$

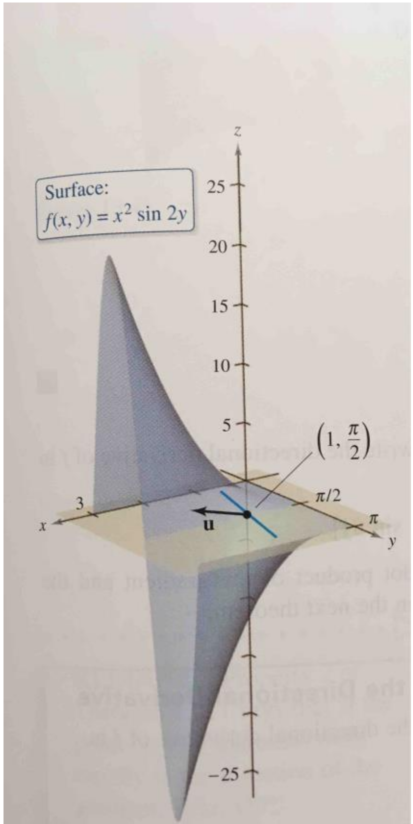


Figure 13.47

Definition of Gradient of a Function of Two Variables

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of f**, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

(The symbol ∇f is read as "del f.") Another notation for the gradient is **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

EXAMPLE 3 **Finding the Gradient of a Function**

Find the gradient of

$$f(x, y) = y \ln x + xy^2$$

at the point $(1, 2)$.

THEOREM 13.10 **Alternative Form of the Directional Derivative**

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

EXAMPLE 4 **Using $\nabla f(x, y)$ to Find a Directional Derivative**

Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

at $\left(-\frac{3}{4}, 0\right)$ in the direction from $P\left(-\frac{3}{4}, 0\right)$ to $Q(0, 1)$.

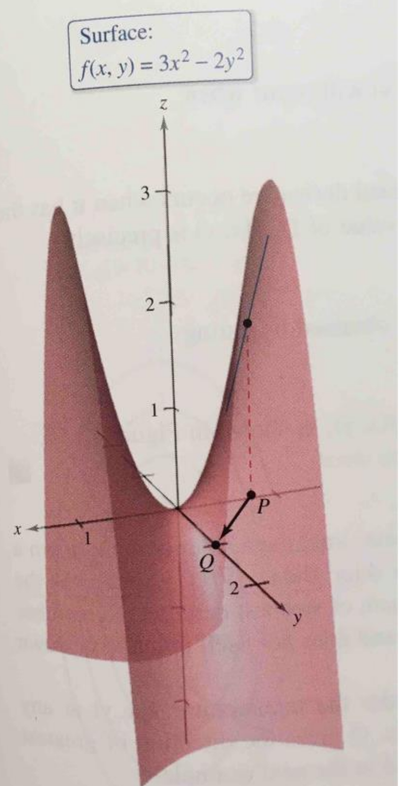


Figure 13.49

THEOREM 13.11 Properties of the Gradient

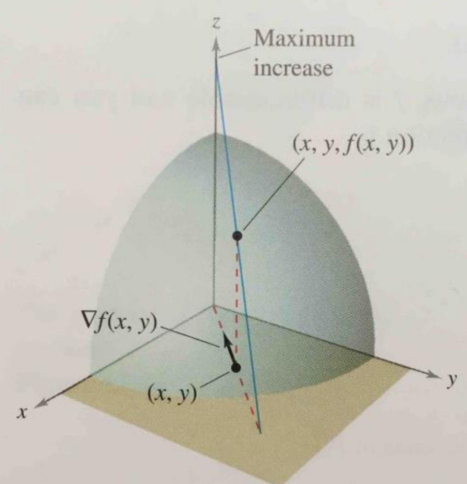
Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is

   $$\|\nabla f(x, y)\|. \qquad \text{Maximum value of } D_{\mathbf{u}} f(x, y)$$

3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is

   $$-\|\nabla f(x, y)\|. \qquad \text{Minimum value of } D_{\mathbf{u}} f(x, y)$$



The gradient of f is a vector in the xy-plane that points in the direction of maximum increase on the surface given by $z = f(x, y)$.

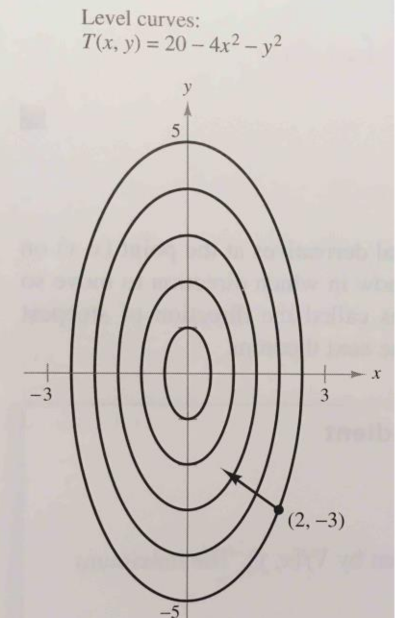
Figure 13.50

EXAMPLE 5 **Finding the Direction of Maximum Increase**

The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?



The direction of most rapid increase in temperature at $(2, -3)$ is given by $-16\mathbf{i} + 6\mathbf{j}$.

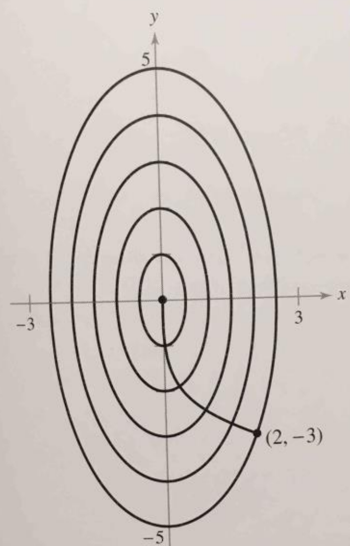
Figure 13.51

EXAMPLE 6 **Finding the Path of a Heat-Seeking Particle**

A heat-seeking particle is located at the point $(2, -3)$ on a metal plate whose temperature at (x, y) is

$$T(x, y) = 20 - 4x^2 - y^2.$$

Find the path of the particle as it continuously moves in the direction of maximum temperature increase.



Path followed by a heat-seeking particle
Figure 13.52

THEOREM 13.12 **Gradient Is Normal to Level Curves**

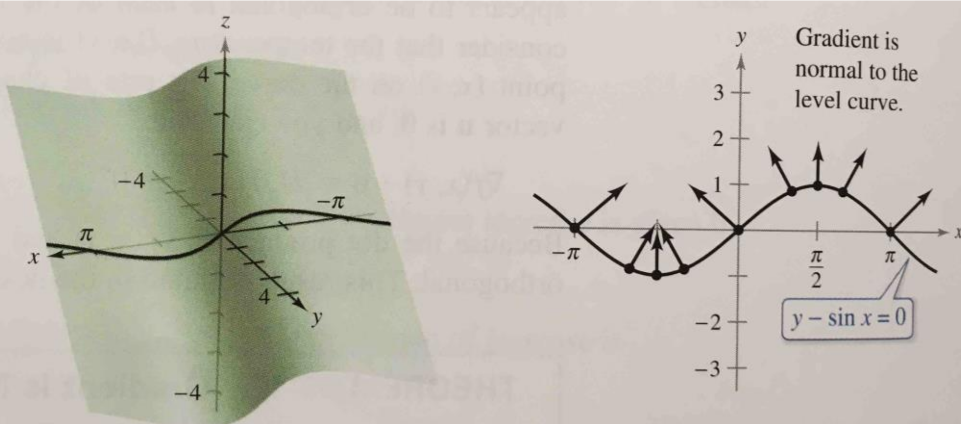
If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

EXAMPLE 7 **Finding a Normal Vector to a Level Curve**

Sketch the level curve corresponding to $c = 0$ for the function given by

$$f(x, y) = y - \sin x$$

and find a normal vector at several points on the curve.



(a) The surface is given by $f(x, y) = y - \sin x$.

(b) The level curve is given by $f(x, y) = 0$.

Figure 13.53

Directional Derivative and Gradient for Three Variables

Let f be a function of x, y, and z, with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of f** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $$\|\nabla f(x, y, z)\|. \qquad \text{Maximum value of } D_{\mathbf{u}} f(x, y, z)$$

4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $$-\|\nabla f(x, y, z)\|. \qquad \text{Minimum value of } D_{\mathbf{u}} f(x, y, z)$$

EXAMPLE 8 Finding the Gradient of a Function

Find $\nabla f(x, y, z)$ for the function

$$f(x, y, z) = x^2 + y^2 - 4z$$

and find the direction of maximum increase of f at the point $(2, -1, 1)$.