Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

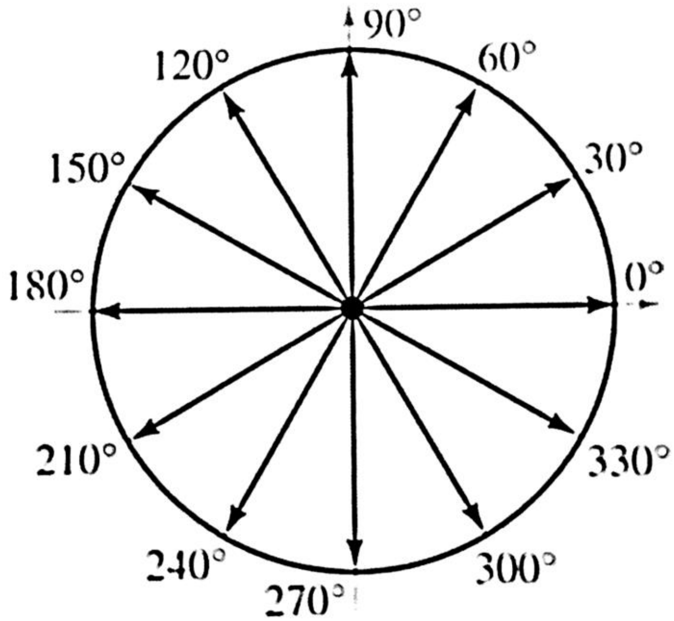
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

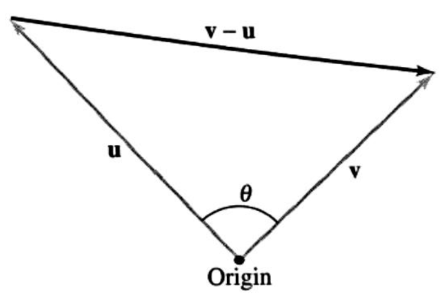
THEOREM 11.4 Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative Property
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive Property
3. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$
4. $\mathbf{0} \cdot \mathbf{v} = 0$
5. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

vectors.





The angle between two vectors
Figure 11.24

THEOREM 11.5 Angle Between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, where $0 \leq \theta \leq \pi$, then

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

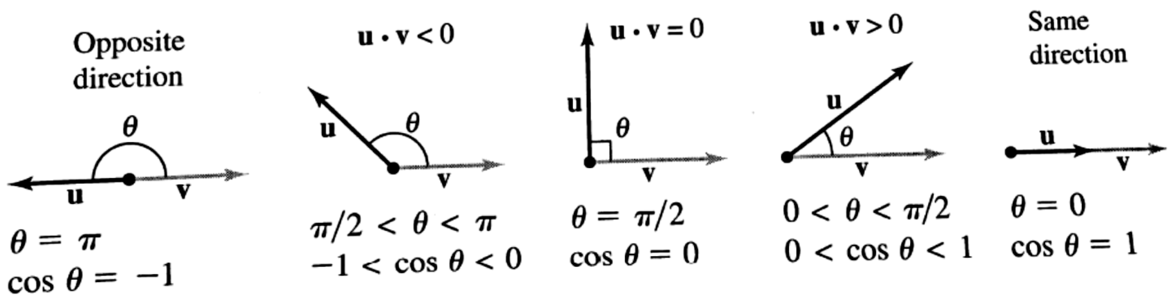


Figure 11.25

EXAMPLE 1 Finding Dot Products

Let $\mathbf{u} = \langle 2, -2\rangle$, $\mathbf{v} = \langle 5, 8\rangle$, and $\mathbf{w} = \langle -4, 3\rangle$.

Definition of Orthogonal Vectors

The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal when $\mathbf{u}\cdot\mathbf{v} = 0$.

EXAMPLE 2 Finding the Angle Between Two Vectors

See LarsonCalculus.com for an interactive version of this type of example.

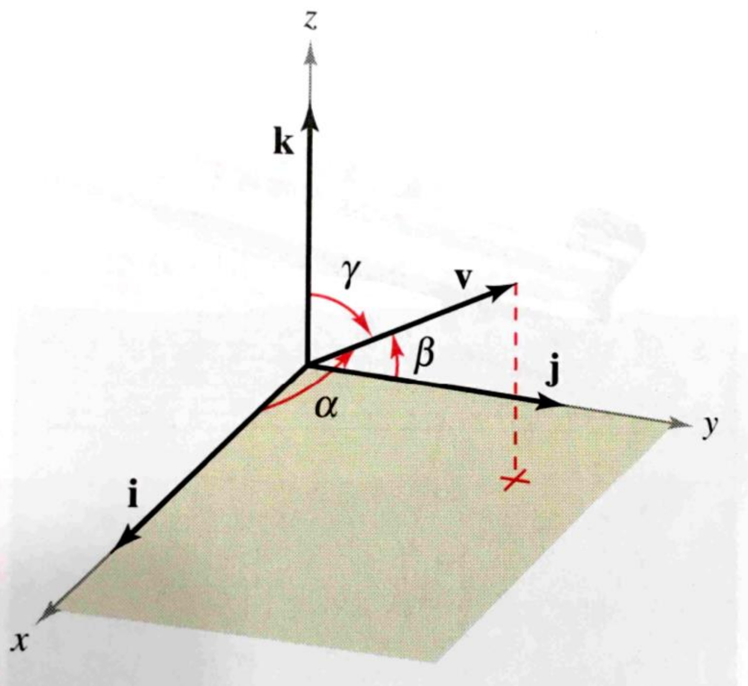
For $\mathbf{u} = \langle 3, -1, 2 \rangle$, $\mathbf{v} = \langle -4, 0, 2 \rangle$, $\mathbf{w} = \langle 1, -1, -2 \rangle$, and $\mathbf{z} = \langle 2, 0, -1 \rangle$, find the angle between each pair of vectors.

a. $\mathbf{u}$ and $\mathbf{v}$ **b.** $\mathbf{u}$ and $\mathbf{w}$ **c.** $\mathbf{v}$ and $\mathbf{z}$

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$ Alternative form of dot product

EXAMPLE 3 Alternative Form of the Dot Product

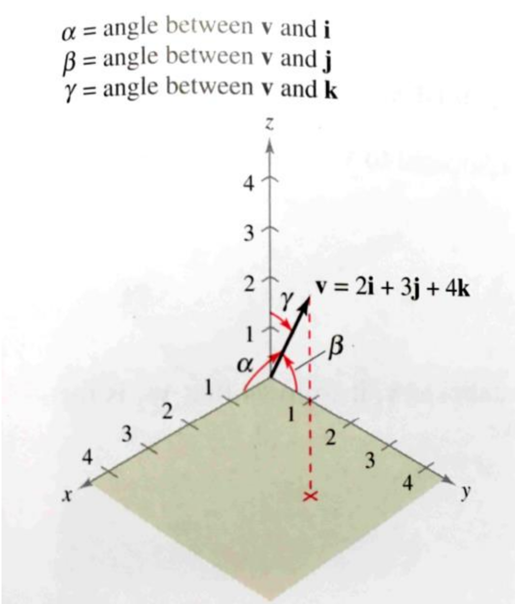
Given that $\|\mathbf{u}\| = 10$, $\|\mathbf{v}\| = 7$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/4$, find $\mathbf{u} \cdot \mathbf{v}$.



Direction angles
Figure 11.26

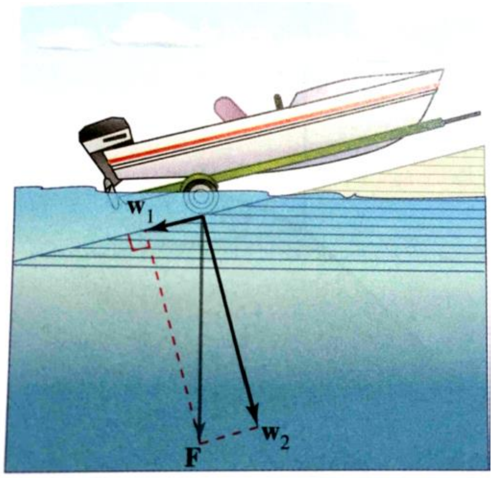
EXAMPLE 4 Finding Direction Angles

Find the direction cosines and angles for the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and show that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

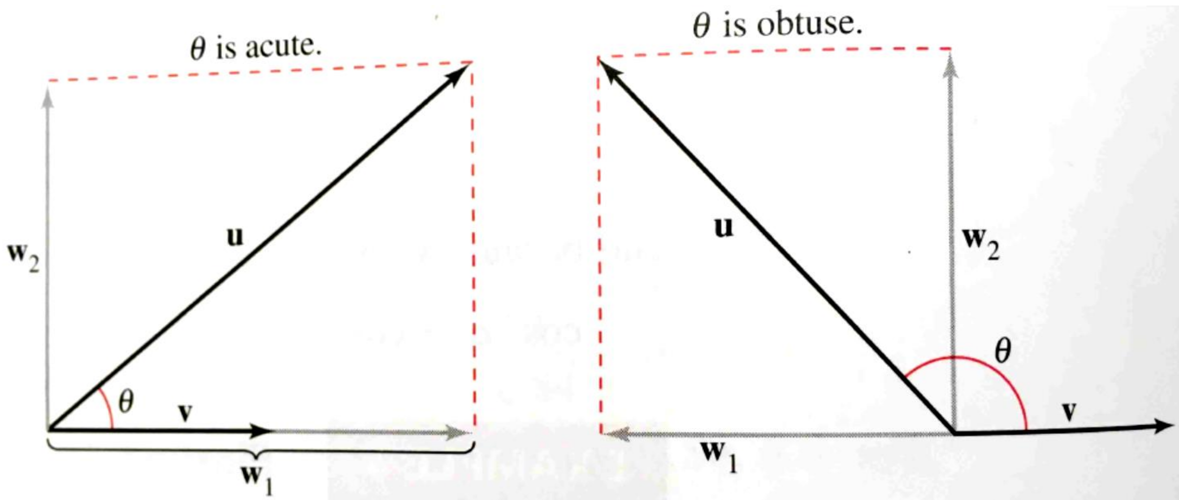


The direction angles of **v**

Figure 11.27



The force due to gravity pulls the boat against the ramp and down the ramp.
Figure 11.28



$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} =$ projection of **u** onto $\mathbf{v} =$ vector component of **u** along **v**
$\mathbf{w}_2 =$ vector component of **u** orthogonal to **v**
Figure 11.29

Definitions of Projection and Vector Components

Let **u** and **v** be nonzero vectors. Moreover, let

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ is parallel to **v** and $\mathbf{w}_2$ is orthogonal to **v**, as shown in Figure 11.29.

1. $\mathbf{w}_1$ is called the **projection of u onto v** or the **vector component of u along v,** and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.
2. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ is called the **vector component of u orthogonal to v.**

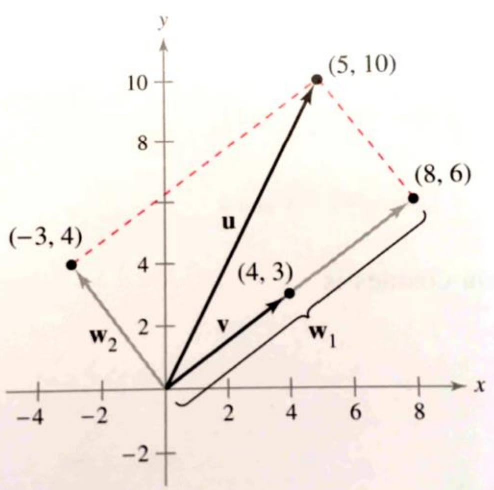
EXAMPLE 5 **Finding a Vector Component of u Orthogonal to v**

Find the vector component of $\mathbf{u} = \langle 5, 10\rangle$ that is orthogonal to $\mathbf{v} = \langle 4, 3\rangle$, given that

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 8, 6\rangle$$

and

$$\mathbf{u} = \langle 5, 10\rangle = \mathbf{w}_1 + \mathbf{w}_2.$$



$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$

Figure 11.30

THEOREM 11.6 Projection Using the Dot Product

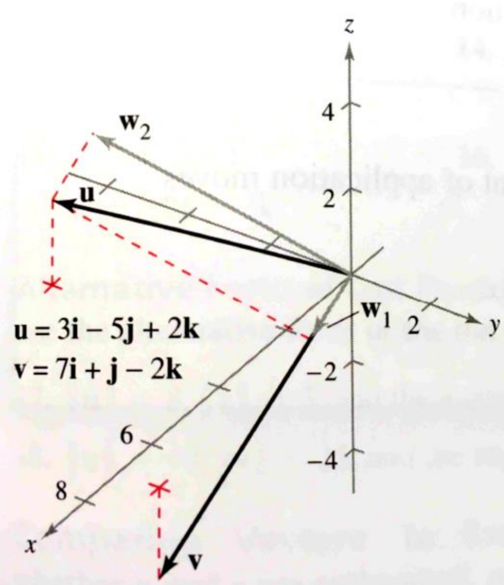
If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

EXAMPLE 6 **Decomposing a Vector into Vector Components**

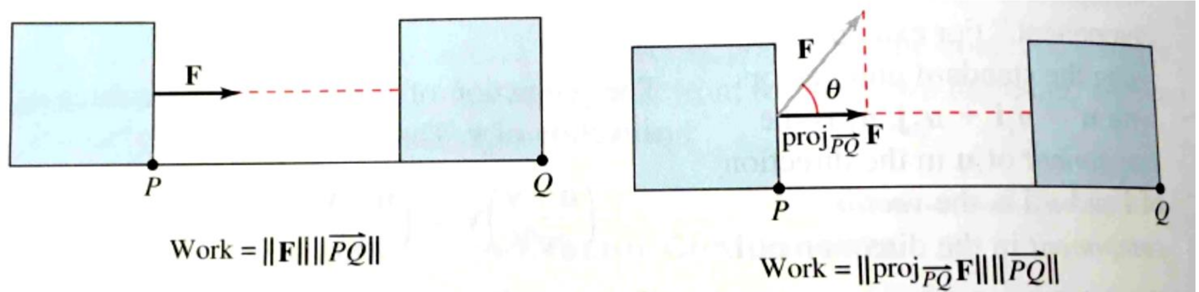
Find the projection of $\mathbf{u}$ onto $\mathbf{v}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ for

$$\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{v} = 7\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$



$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$

Figure 11.31



(a) Force acts along the line of motion. **(b)** Force acts at angle θ with the line of motion.

Figure 11.33

Definition of Work

The work W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is one of the following.

1. $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\|$ Projection form
2. $W = \mathbf{F} \cdot \overrightarrow{PQ}$ Dot product form

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EXAMPLE 8 Finding Work

To close a sliding door, a person pulls on a rope with a constant force of 50 pounds at a constant angle of 60°, as shown in Figure 11.34. Find the work done in moving the door 12 feet to its closed position.

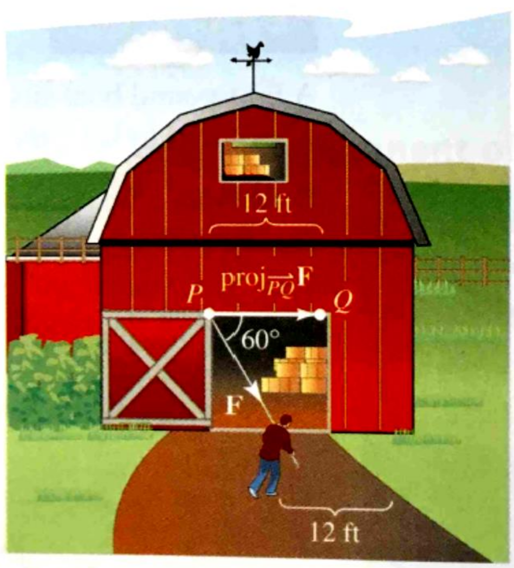

Figure 11.34