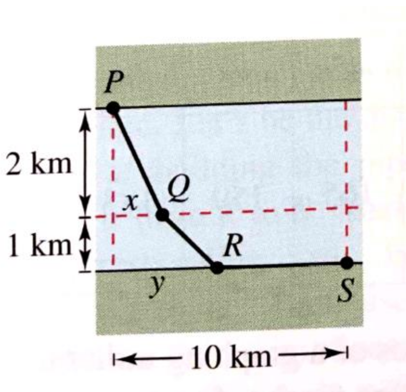
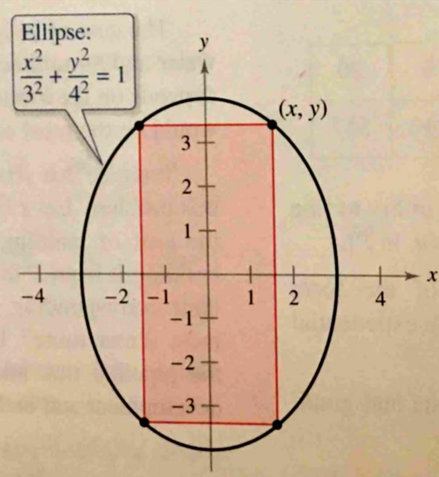
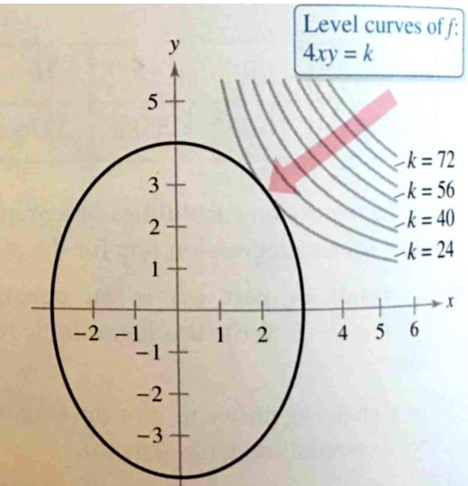
. **Minimum Cost** A water line is to be built from point P to point S and must pass through regions where construction costs differ (see figure). The cost per kilometer (in dollars) is $3k$ from P to Q, $2k$ from Q to R, and k from R to S. Find x and y such that the total cost C will be minimized.





Objective function: $f(x, y) = 4xy$

Figure 13.78



Constraint: $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$

Figure 13.79

THEOREM 13.19 Lagrange's Theorem

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Method of Lagrange Multipliers

Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use these steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The greatest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the least value yields the minimum of f subject to the constraint $g(x, y) = c$.

EXAMPLE 1 **Using a Lagrange Multiplier with One Constraint**

Find the maximum value of $f(x, y) = 4xy$, where $x > 0$ and $y > 0$, subject to the constraint $(x^2/3^2) + (y^2/4^2) = 1$.

14.1

$$\int_{h_1(y)}^{h_2(y)} f_x(x, y)\, dx = f(x, y)\Big]_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y) \quad \text{With respect to } x$$

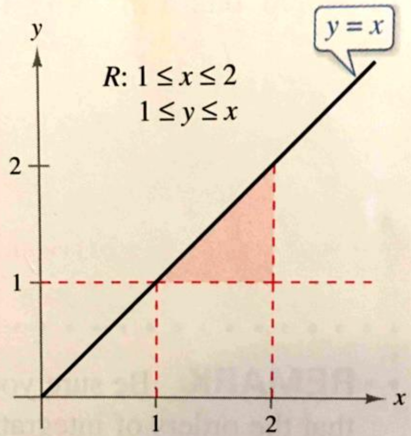
$$\int_{g_1(x)}^{g_2(x)} f_y(x, y)\, dy = f(x, y)\Big]_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)) \quad \text{With respect to } y$$

EXAMPLE 1 Integrating with Respect to *y*

Evaluate $\displaystyle\int_1^x (2x^2y^{-2} + 2y)\, dy$.

EXAMPLE 2 The Integral of an Integral

Evaluate $\displaystyle\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y)\, dy\right] dx$.



The region of integration for

$$\int_1^2 \int_1^x f(x, y)\, dy\, dx$$

Figure 14.1

Area of a Region in the Plane

1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then the area of R is

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx.$$ Figure 14.2 (vertically simple)

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then the area of R is

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$ Figure 14.3 (horizontally simple)

EXAMPLE 3 **The Area of a Rectangular Region**

Use an iterated integral to represent the area of the rectangle shown in Figure 14.4.

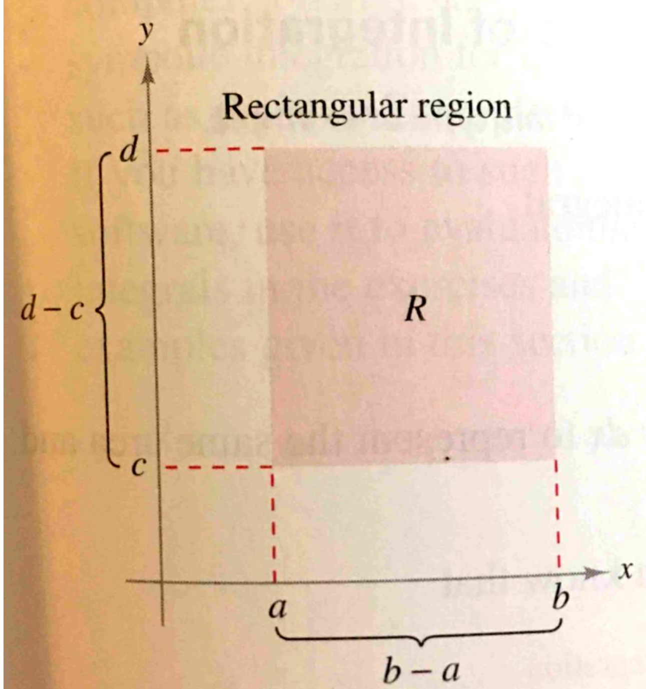


Figure 14.4

EXAMPLE 4 **Finding Area by an Iterated Integral**

Use an iterated integral to find the area of the region bounded by the graphs of

$$f(x) = \sin x$$ Sine curve forms upper boundary.

and

$$g(x) = \cos x$$ Cosine curve forms lower boundary.

between $x = \pi/4$ and $x = 5\pi/4$.

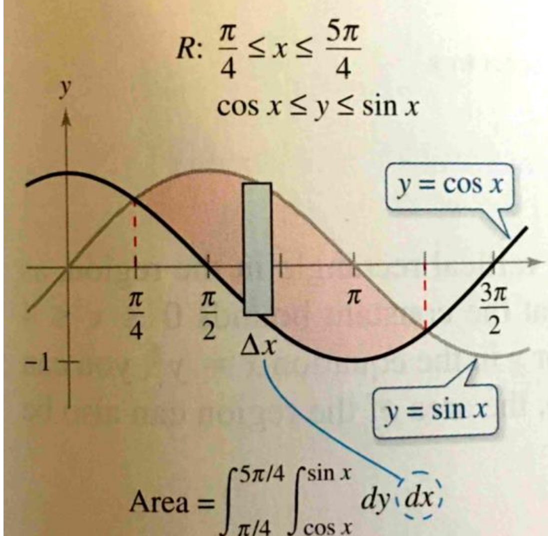


Figure 14.5

EXAMPLE 5 **Comparing Different Orders of Integration**

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the region whose area is represented by the integral

$$\int_0^2 \int_{y^2}^4 dx\, dy.$$

Then find another iterated integral using the order $dy\,dx$ to represent the same area and show that both integrals yield the same value.

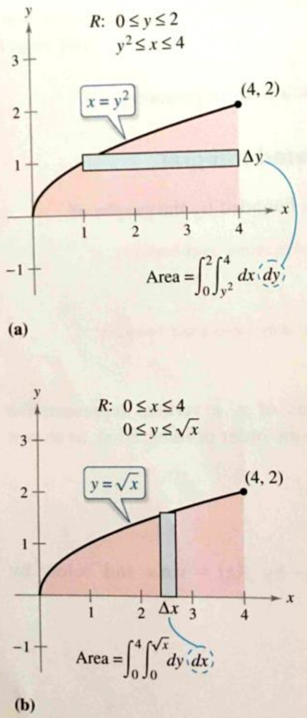


Figure 14.6

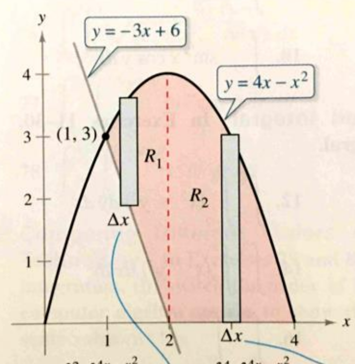
EXAMPLE 6 **An Area Represented by Two Iterated Integrals**

Find the area of the region R that lies below the parabola

$$y = 4x - x^2$$ Parabola forms upper boundary.

above the x-axis, and above the line

$$y = -3x + 6.$$ Line and x-axis form lower boundary.



$$\text{Area} = \int_1^2 \int_{-3x+6}^{4x-x^2} dy\, dx + \int_2^4 \int_0^{4x-x^2} dy\, dx$$

Figure 14.7