THEOREM 9.17 Ratio Test

Let Σa_n be a series with nonzero terms.

1. The series Σa_n converges absolutely when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. The series Σa_n diverges when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$.
3. The Ratio Test is inconclusive when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

EXAMPLE 1 Using the Ratio Test

Determine the convergence or divergence of

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

EXAMPLE 2 Using the Ratio Test

Determine whether each series converges or diverges.

a. $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **b.** $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

EXAMPLE 3 A Failure of the Ratio Test

See LarsonCalculus.com for an interactive version of this ty

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}.$$

THEOREM 9.18 Root Test

1. The series $\Sigma\, a_n$ converges absolutely when $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. The series $\Sigma\, a_n$ diverges when $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

EXAMPLE 4 Using the Root Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}.$$

GUIDELINES FOR TESTING A SERIES FOR CONVERGENCE OR DIVERGENCE

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

EXAMPLE 5 Applying the Strategies for Testing Series

Determine the convergence or divergence of each series.

a. $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$

b. $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$

c. $\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$

e. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$

f. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

g. $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

nth-Term	$\sum_{n=1}^{\infty} a_n$	
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$0 < \lvert r \rvert < 1$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \le a_n$ and $\lim_{n\to\infty} a_n = 0$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \ge 0$	$\int_1^{\infty} f(x)\, dx$ converges
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1$
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges