THEOREM 3.3 Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

EXAMPLE 1 Illustrating Rolle's Theorem

Find the two x-intercepts of

$$f(x) = x^2 - 3x + 2$$

and show that $f'(x) = 0$ at some point between the two x-intercepts.

EXAMPLE 2 Illustrating Rolle's Theorem

Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval $(-2, 2)$ such that $f'(c) = 0$.

THEOREM 3.4 The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

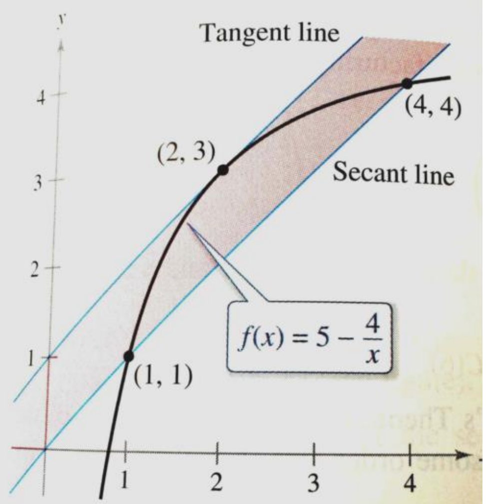
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

EXAMPLE 3 Finding a Tangent Line

See LarsonCalculus.com for an interactive version of this type of example.

For $f(x) = 5 - (4/x)$, find all values of c in the open interval $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

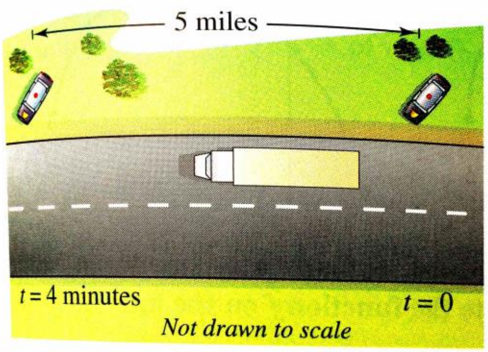


The tangent line at (2, 3) is parallel to the secant line through (1, 1) and (4, 4).

Figure 3.13

EXAMPLE 4 Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 3.14. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes.



At some time t, the instantaneous velocity is equal to the average velocity over 4 minutes.

Figure 3.14

51. Vertical Motion The height of an object t seconds after it is dropped from a height of 300 meters is

$$s(t) = -4.9t^2 + 300.$$

(a) Find the average velocity of the object during the first 3 seconds.

(b) Use the Mean Value Theorem to verify that at some time during the first 3 seconds of fall, the instantaneous velocity equals the average velocity. Find that time.

52. Sales A company introduces a new product for which the number of units sold S is

$$S(t) = 200\left(5 - \frac{9}{2 + t}\right)$$

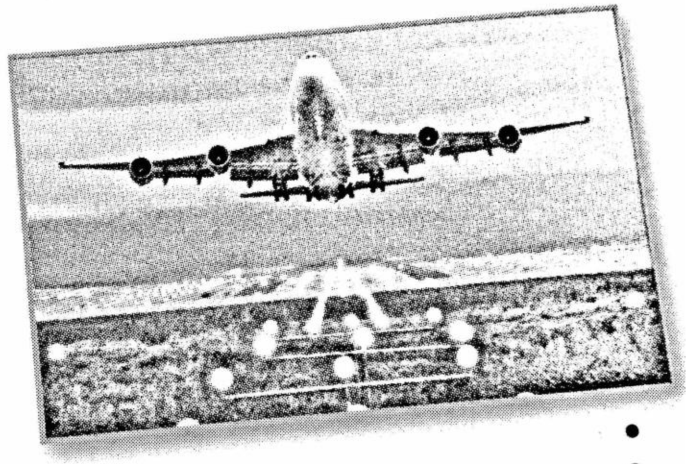
where t is the time in months.

(a) Find the average rate of change of $S(t)$ during the first year.

(b) During what month of the first year does $S'(t)$ equal the average rate of change?

57. Speed

A plane begins its take-off at 2:00 P.M. on a 2500-mile flight. After 5.5 hours, the plane arrives at its destination. Explain why there are at least two times during the flight when the speed of the plane is 400 miles per hour.



58. Temperature When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F, its core temperature is 1500°F. Five hours later, the core temperature is 390°F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222°F per hour.

Using Rolle's Theorem **In Exercises 9–22, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.**

9. $f(x) = -x^2 + 3x, \quad [0, 3]$

10. $f(x) = x^2 - 8x + 5, \quad [2, 6]$

11. $f(x) = (x - 1)(x - 2)(x - 3), \quad [1, 3]$

12. $f(x) = (x - 4)(x + 2)^2, \quad [-2, 4]$

13. $f(x) = x^{2/3} - 1, \quad [-8, 8]$ **14.** $f(x) = 3 - |x - 3|, \quad [0, 6]$

Solutions 9-14

9. $f(x) = -x^2 + 3x, \quad [0, 3]$

$f(0) = -(0)^2 + 3(0)$

$f(3) = -(3)^2 + 3(3) = 0$

f is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Rolle's Theorem applies.

$f'(x) = -2x + 3 = 0$

$-2x = -3 \Rightarrow x = \frac{3}{2}$

c-value: $\frac{3}{2}$

10. $f(x) = x^2 - 8x + 5, [2, 6]$

$f(2) = 4 - 16 + 5 = -7$

$f(6) = 36 - 48 + 5 = -7$

f is continuous on $[2, 6]$ and differentiable on $(2, 6)$. Rolle's Theorem applies.

$f'(x) = 2x - 8 = 0$

$2x = 8 \Rightarrow x = 4$

c-value: 4

11. $f(x) = (x - 1)(x - 2)(x - 3), [1, 3]$

$f(1) = (1 - 1)(1 - 2)(1 - 3) = 0$

$f(3) = (3 - 1)(3 - 2)(3 - 3) = 0$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$. Rolle's Theorem applies.

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11 = 0$

$$x = \frac{6 \pm \sqrt{3}}{3}$$

c-values: $\frac{6 - \sqrt{3}}{3}, \frac{6 + \sqrt{3}}{3}$

12. $f(x) = (x - 4)(x + 2)^2, [-2, 4]$

$f(-2) = (-2 - 4)(-2 + 2)^2 = 0$

$f(4) = (4 - 4)(4 + 2)^2 = 0$

f is continuous on $[-2, 4]$. f is differentiable on $(-2, 4)$. Rolle's Theorem applies.

$f(x) = (x - 4)(x^2 + 4x + 4) = x^3 - 12x - 16$

$f'(x) = 3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4 \Rightarrow x = \pm 2$

(Note: $x = -2$ is not in the interval.)

c-value: 2

13. $f(x) = x^{2/3} - 1, [-8, 8]$

$f(-8) = (-8)^{2/3} - 1 = 3$

$f(8) = (8)^{2/3} - 1 = 3$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ because $f'(0)$ does not exist. Rolle's Theorem does not apply.

14. $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ because $f'(3)$ does not exist. Rolle's Theorem does not apply.