Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series.** More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at c,** where c is a constant.

EXAMPLE 1 Power Series

a. The following power series is centered at 0.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

b. The following power series is centered at -1.

$$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+$$

c. The following power series is centered at 1.

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n = (x-1) + \frac{1}{2}(x-1)^2 +$$

The domain of a power series has only three basic forms: a single point, an interval centered at c, or the entire real number line.
Figure 9.17

THEOREM 9.20 Convergence of a Power Series

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for

$$|x - c| < R$$

and diverges for

$$|x - c| > R.$$

3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, then the radius of convergence is $R = 0$. If the series converges for all x, then the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 2 Finding the Radius of Convergence

Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$.

EXAMPLE 3 Finding the Radius of Convergence

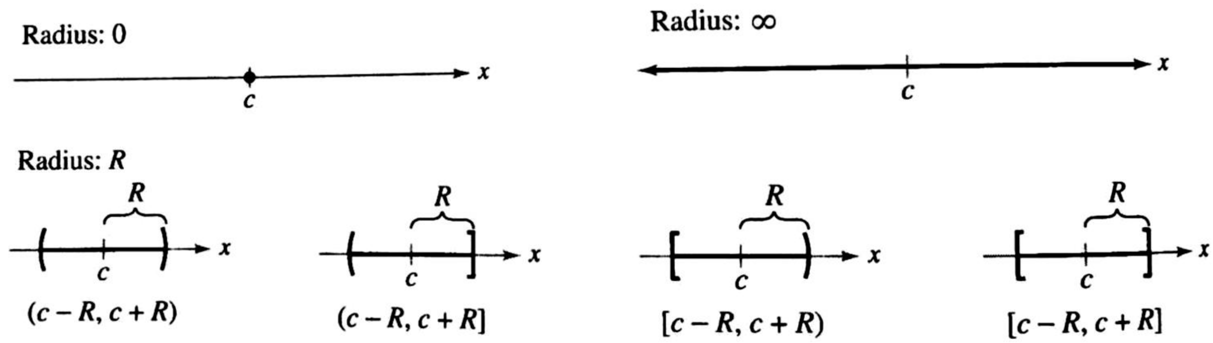
Find the radius of convergence of

$$\sum_{n=0}^{\infty} 3(x-2)^n.$$

EXAMPLE 4 Finding the Radius of Convergence

Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$



Intervals of convergence
Figure 9.18

EXAMPLE 5 Finding the Interval of Convergence

See LarsonCalculus.com for an interactive version of this type of example.

Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

EXAMPLE 6 Finding the Interval of Convergence

Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n(x+1)^n}{2^n}$.

EXAMPLE 7 Finding the Interval of Convergence

Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

THEOREM 9.21 Properties of Functions Defined by Power Series

If the function

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$
$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

has a radius of convergence of $R > 0$, then, on the interval

$$(c - R, c + R)$$

f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$
$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

2. $\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$
$$= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence,* however, may differ as a result of the behavior at the endpoints.

EXAMPLE 8 **Intervals of Convergence for $f(x)$, $f'(x)$, and $\int f(x)\,dx$**

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

Find the interval of convergence for each of the following.

a. $\int f(x)\,dx$ **b.** $f(x)$ **c.** $f'(x)$