EXAMPLE 1 Two Rates That Are Related

The variables x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

EXAMPLE 2 Ripples in a Pond

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in Figure 2.34. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour.	$x =$ distance traveled $\frac{dx}{dt} = 50$ mi/h when $t = 1$
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.	$V =$ volume of water in pool $\frac{dV}{dt} = 10 \text{ m}^3/\text{h}$
A gear is revolving at a rate of 25 revolutions per minute (1 revolution $= 2\pi$ rad).	$\theta =$ angle of revolution $\frac{d\theta}{dt} = 25(2\pi)$ rad/min
A population of bacteria is increasing at a rate of 2000 per hour.	$x =$ number in population $\frac{dx}{dt} = 2000$ bacteria per hour

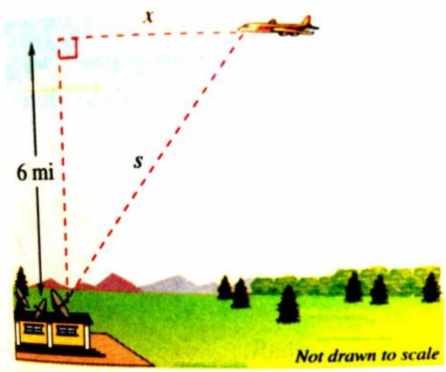
EXAMPLE 3 An Inflating Balloon

Air is being pumped into a spherical balloon (see Figure 2.35) at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

EXAMPLE 4 **The Speed of an Airplane Tracked by Radar**

See LarsonCalculus.com for an interactive version of this type of example.

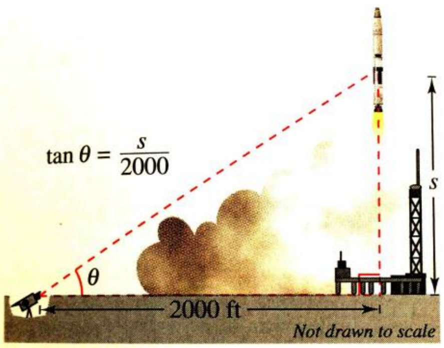
An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in Figure 2.36. The distance s is decreasing at a rate of 400 miles per hour when $s = 10$ miles. What is the speed of the plane?



An airplane is flying at an altitude of 6 miles, s miles from the station.
Figure 2.36

EXAMPLE 5 A Changing Angle of Elevation

Find the rate of change in the angle of elevation of the camera shown in Figure 2.37 at 10 seconds after lift-off.

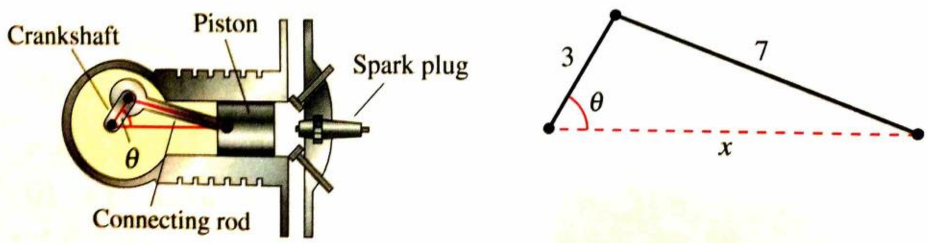


A television camera at ground level is filming the lift-off of a rocket that is rising vertically according to the position equation $s = 50t^2$, where s is measured in feet and t is measured in seconds. The camera is 2000 feet from the launch pad.

Figure 2.37

EXAMPLE 6 The Velocity of a Piston

In the engine shown in Figure 2.38, a 7-inch connecting rod is fastened to a crank of radius 3 inches. The crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute. Find the velocity of the piston when $\theta = \pi/3$.



The velocity of a piston is related to the angle of the crankshaft.
Figure 2.38

Moving Point **In Exercises 5–8, a point is moving along the graph of the given function at the rate dx/dt. Find dy/dt for the given values of x.**

5. $y = 2x^2 + 1$; $\frac{dx}{dt} = 2$ centimeters per second

(a) $x = -1$ (b) $x = 0$ (c) $x = 1$

6. $y = \frac{1}{1 + x^2}$; $\frac{dx}{dt} = 6$ inches per second

(a) $x = -2$ (b) $x = 0$ (c) $x = 2$

7. $y = \tan x$; $\frac{dx}{dt} = 3$ feet per second

(a) $x = -\frac{\pi}{3}$ (b) $x = -\frac{\pi}{4}$ (c) $x = 0$

8. $y = \cos x$; $\frac{dx}{dt} = 4$ centimeters per second

(a) $x = \frac{\pi}{6}$ (b) $x = \frac{\pi}{4}$ (c) $x = \frac{\pi}{3}$

WRITING ABOUT CONCEPTS

9. Related Rates Consider the linear function

$y = ax + b.$

If x changes at a constant rate, does y change at a constant rate? If so, does it change at the same rate as x? Explain.

10. Related Rates In your own words, state the guidelines for solving related-rate problems.

11. Area The radius r of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of change of the area when (a) $r = 8$ centimeters and (b) $r = 32$ centimeters.

12. Area The included angle of the two sides of constant equal length s of an isosceles triangle is θ.

(a) Show that the area of the triangle is given by $A = \frac{1}{2}s^2 \sin \theta$.

(b) The angle θ is increasing at the rate of $\frac{1}{2}$ radian per minute. Find the rates of change of the area when $\theta = \pi/6$ and $\theta = \pi/3$.

(c) Explain why the rate of change of the area of the triangle is not constant even though $d\theta/dt$ is constant.

13. Volume The radius r of a sphere is increasing at a rate of 3 inches per minute.

(a) Find the rates of change of the volume when $r = 9$ inches and $r = 36$ inches.

(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

14. Volume A spherical balloon is inflated with gas at the rate of 800 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 30 centimeters and (b) 60 centimeters?

15. Volume All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the volume changing when each edge is (a) 2 centimeters and (b) 10 centimeters?

16. Surface Area All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the surface area changing when each edge is (a) 2 centimeters and (b) 10 centimeters?

17. Volume At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high? (*Hint:* The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

18. Depth A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. Water is flowing into the tank at a rate of 10 cubic feet per minute. Find the rate of change of the depth of the water when the water is 8 feet deep.

19. Depth A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end (see figure). Water is being pumped into the pool at $\frac{1}{4}$ cubic meter per minute, and there is 1 meter of water at the deep end.

7. $y = \tan x, \dfrac{dx}{dt} = 3$

$\dfrac{dy}{dt} = \sec^2 x \cdot \dfrac{dx}{dt} = \sec^2 x(3) = 3\sec^2 x$

(a) When $x = -\dfrac{\pi}{3}$:

$\dfrac{dy}{dt} = 3\sec^2\left(-\dfrac{\pi}{3}\right) = 3(2)^2 = 12$ ft/sec

(b) When $x = -\dfrac{\pi}{4}$:

$\dfrac{dy}{dt} = 3\sec^2\left(-\dfrac{\pi}{4}\right) = 3\left(\sqrt{2}\right)^2 = 6$ ft/sec

(c) When $x = 0$:

$\dfrac{dy}{dt} = 3\sec^2(0) = 3$ ft/sec

8. $y = \cos x, \dfrac{dx}{dt} = 4$

$\dfrac{dy}{dt} = -\sin x \cdot \dfrac{dx}{dt} = -\sin x(4) = -4\sin x$

(a) When $x = \dfrac{\pi}{6}$:

$\dfrac{dy}{dt} = -4\sin\left(\dfrac{\pi}{6}\right) = -4\left(\dfrac{1}{2}\right) = -2$ cm/sec

(b) When $x = \dfrac{\pi}{4}$:

$\dfrac{dy}{dt} = -4\sin\left(\dfrac{\pi}{4}\right) = -4\left(\dfrac{\sqrt{2}}{2}\right) = -2\sqrt{2}$ cm/sec

(c) When $x = \dfrac{\pi}{3}$:

$\dfrac{dy}{dt} = -4\sin\left(\dfrac{\pi}{3}\right) = -4\left(\dfrac{\sqrt{3}}{2}\right) = -2\sqrt{3}$ cm/sec

9. Yes, y changes at a constant rate.

$\dfrac{dy}{dt} = a \cdot \dfrac{dx}{dt}$

No, the rate dy/dt is a multiple of dx/dt.

10. Answers will vary. See page 149.

11. $A = \pi r^2$

$\dfrac{dr}{dt} = 4$

$\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$

(a) When $r = 8, \dfrac{dA}{dt} = 2\pi(8)(4) = 64\pi$ cm²/min.

(b) When $r = 32, \dfrac{dA}{dt} = 2\pi(32)(4) = 256\pi$ cm²/min.

12. (a) $\sin\dfrac{\theta}{2} = \dfrac{(1/2)b}{s} \Rightarrow b = 2s\sin\dfrac{\theta}{2}$

$\cos\dfrac{\theta}{2} = \dfrac{h}{s} \Rightarrow h = s\cos\dfrac{\theta}{2}$

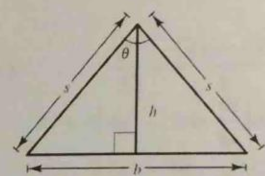
$A = \dfrac{1}{2}bh = \dfrac{1}{2}\left(2s\sin\dfrac{\theta}{2}\right)\left(s\cos\dfrac{\theta}{2}\right)$

$= \dfrac{s^2}{2}\left(2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\right) = \dfrac{s^2}{2}\sin\theta$

(b) $\dfrac{dA}{dt} = \dfrac{s^2}{2}\cos\theta\dfrac{d\theta}{dt}$ where $\dfrac{d\theta}{dt} = \dfrac{1}{2}$ rad/min.

When $\theta = \dfrac{\pi}{6}, \dfrac{dA}{dt} = \dfrac{s^2}{2}\left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{\sqrt{3}s^2}{8}$.

When $\theta = \dfrac{\pi}{3}, \dfrac{dA}{dt} = \dfrac{s^2}{2}\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{s^2}{8}$.



(c) If s and $\dfrac{d\theta}{dt}$ is constant, $\dfrac{dA}{dt}$ is proportional to $\cos\theta$.

13. $V = \dfrac{4}{3}\pi r^3$

$\dfrac{dr}{dt} = 3$

$\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$

(a) When $r = 9$,

$\dfrac{dV}{dt} = 4\pi(9)^2(3) = 972\pi$ in.³/min.

When $r = 36$,

$\dfrac{dV}{dt} = 4\pi(36)^2(3) = 15{,}552\pi$ in.³/min.

(b) If dr/dt is constant, dV/dt is proportional to r^2.

14. $V = \dfrac{4}{3}\pi r^3, \dfrac{dV}{dt} = 800$

$\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{1}{4\pi r^2}\left(\dfrac{dV}{dt}\right) = \dfrac{1}{4\pi r^2}(800)$

(a) When $r = 30$,

$\dfrac{dr}{dt} = \dfrac{1}{4\pi(30)^2}(800) = \dfrac{2}{9\pi}$ cm/min.

(b) When $r = 60$,

$\dfrac{dr}{dt} = \dfrac{1}{4\pi(60)^2}(800) = \dfrac{1}{18\pi}$ cm/min.

15. $V = x^3$

$\frac{dx}{dt} = 6$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

(a) When $x = 2$,

$\frac{dV}{dt} = 3(2)^2(6) = 72 \text{ cm}^3/\text{sec}.$

(b) When $x = 10$,

$\frac{dV}{dt} = 3(10)^2(6) = 1800 \text{ cm}^3/\text{sec}.$

16. $s = 6x^2$

$\frac{dx}{dt} = 6$

$\frac{ds}{dt} = 12x \frac{dx}{dt}$

(a) When $x = 2$,

$\frac{ds}{dt} = 12(2)(6) = 144 \text{ cm}^2/\text{sec}.$

(b) When $x = 10$,

$\frac{ds}{dt} = 12(10)(6) = 720 \text{ cm}^2/\text{sec}.$

17. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{9}{4}h^2\right)h$ [because $2r = 3h$]

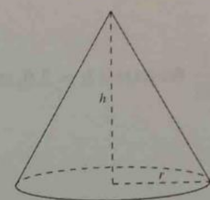
$= \frac{3\pi}{4}h^3$

$\frac{dV}{dt} = 10$

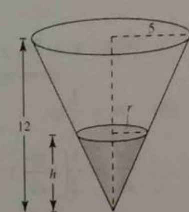
$\frac{dV}{dt} = \frac{9\pi}{4}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$

When $h = 15$,

$\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi}$ ft/min.



18. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144}h^3 = \frac{25\pi}{3(144)}h^3$ (By similar triangles, $\frac{r}{5} = \frac{h}{12} \Rightarrow r = \frac{5}{12}h.$)

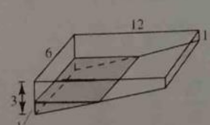


$\frac{dV}{dt} = 10$

$\frac{dV}{dt} = \frac{25\pi}{144}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \left(\frac{144}{25\pi h^2}\right)\frac{dV}{dt}$

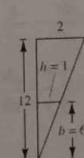
When $h = 8$, $\frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi}$ ft/min.

19.



(a) Total volume of pool $= \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1 m of water $= \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$ (see similar triangle diagram)



% pool filled $= \frac{18}{144}(100\%) = 12.5\%$

(b) Because for $0 \le h \le 2$, $b = 6h$, you have

$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$

$\frac{dV}{dt} = 36h\frac{dh}{dt} = \frac{1}{4} \Rightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144}$ m/min.