Definition of Vector-Valued Function

A function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \qquad \text{Plane}$$

or

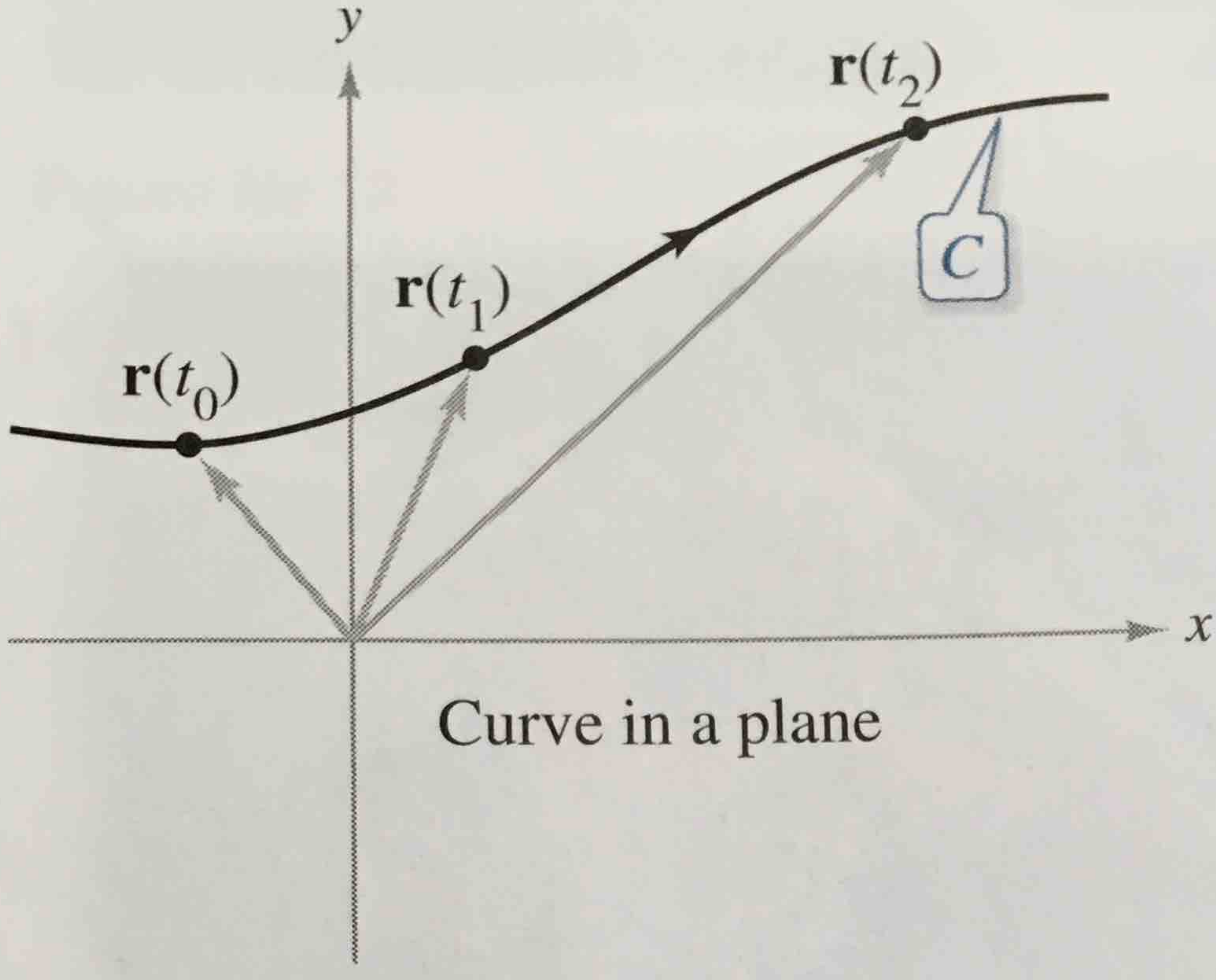
$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad \text{Space}$$

is a **vector-valued function,** where the **component functions** f, g, and h are real-valued functions of the parameter t. Vector-valued functions are sometimes denoted as

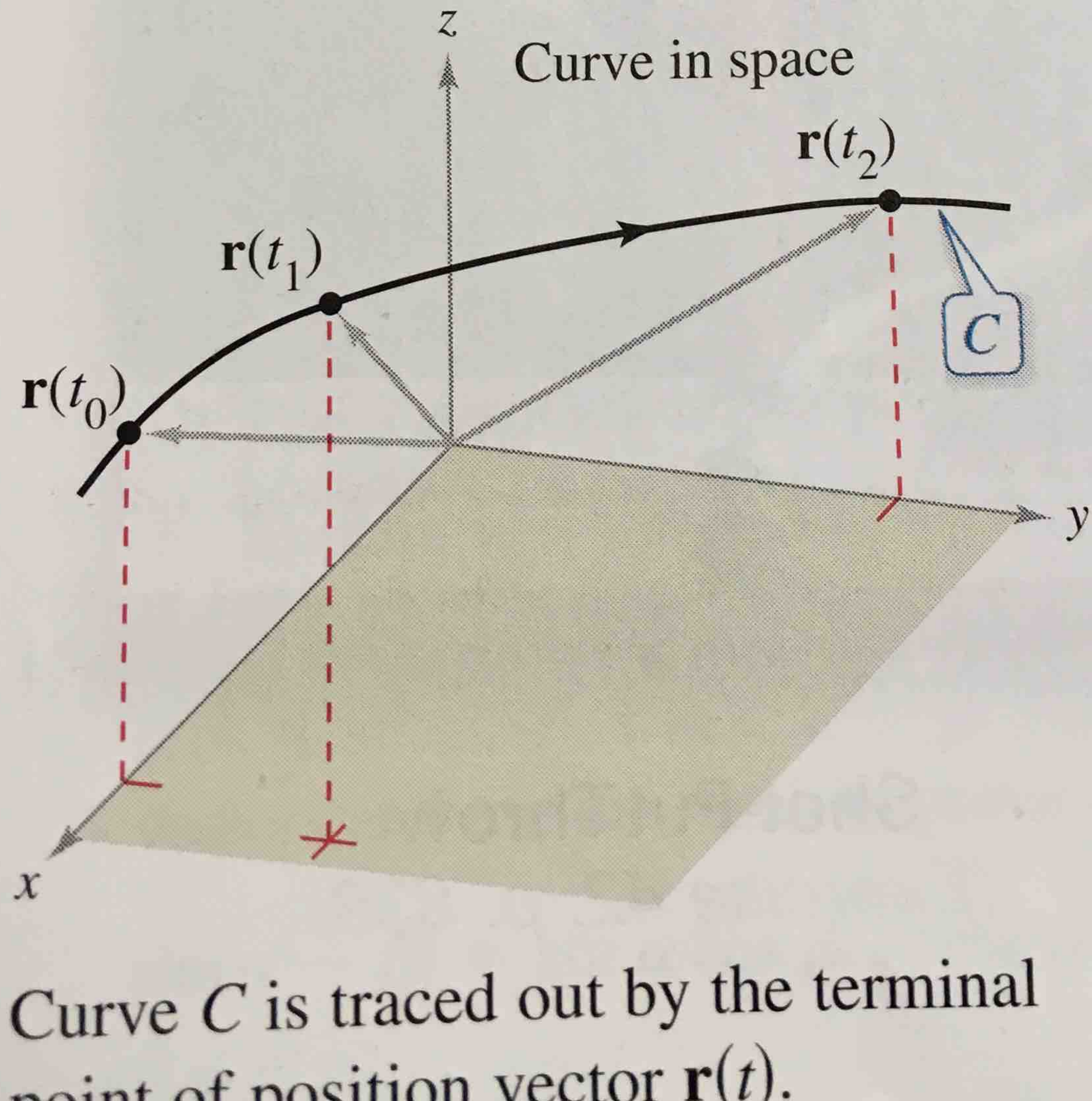
$$\mathbf{r}(t) = \langle f(t), g(t) \rangle \qquad \text{Plane}$$

or

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle. \qquad \text{Space}$$



Curve in a plane



Curve in space

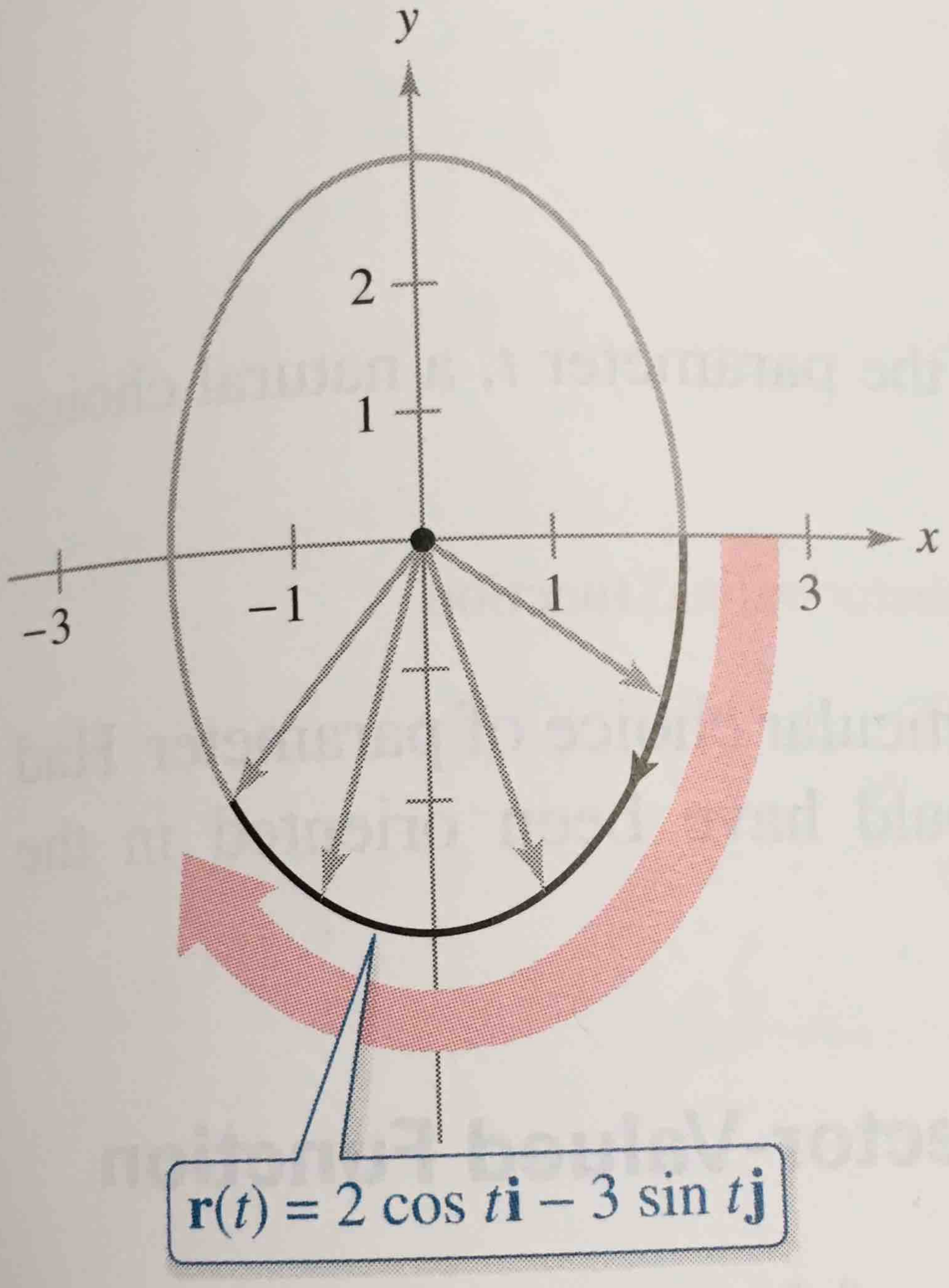
Curve C is traced out by the terminal point of position vector $\mathbf{r}(t)$.

Figure 12.1

EXAMPLE 1 Sketching a Plane Curve

Sketch the plane curve represented by the vector-valued function

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi.$$ Vector-valued function



The ellipse is traced clockwise as t increases from 0 to 2π.

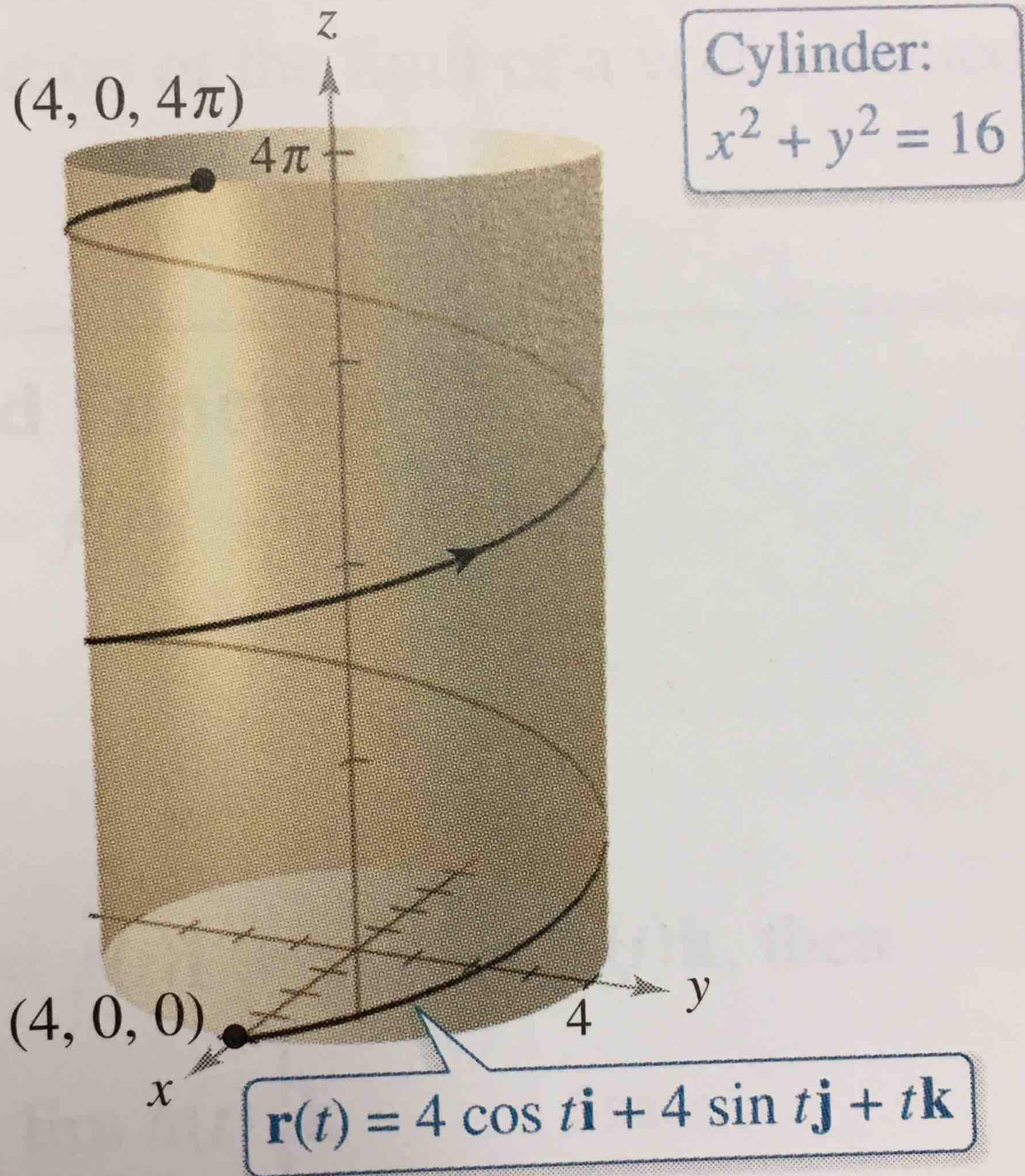
Figure 12.2

EXAMPLE 2 Sketching a Space Curve

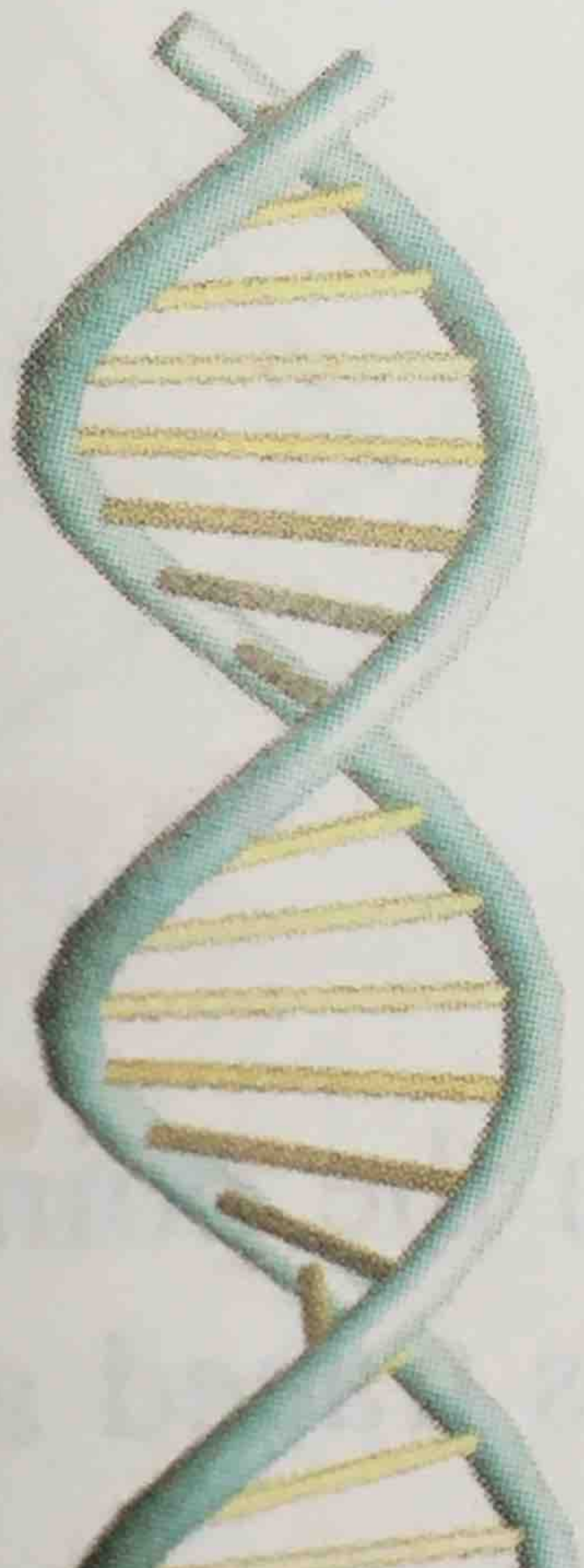
See LarsonCalculus.com for an interactive version of this type of example.

Sketch the space curve represented by the vector-valued function

$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 4\pi.$$ Vector-valued function



As t increases from 0 to 4π, two spirals on the helix are traced out.
Figure 12.3



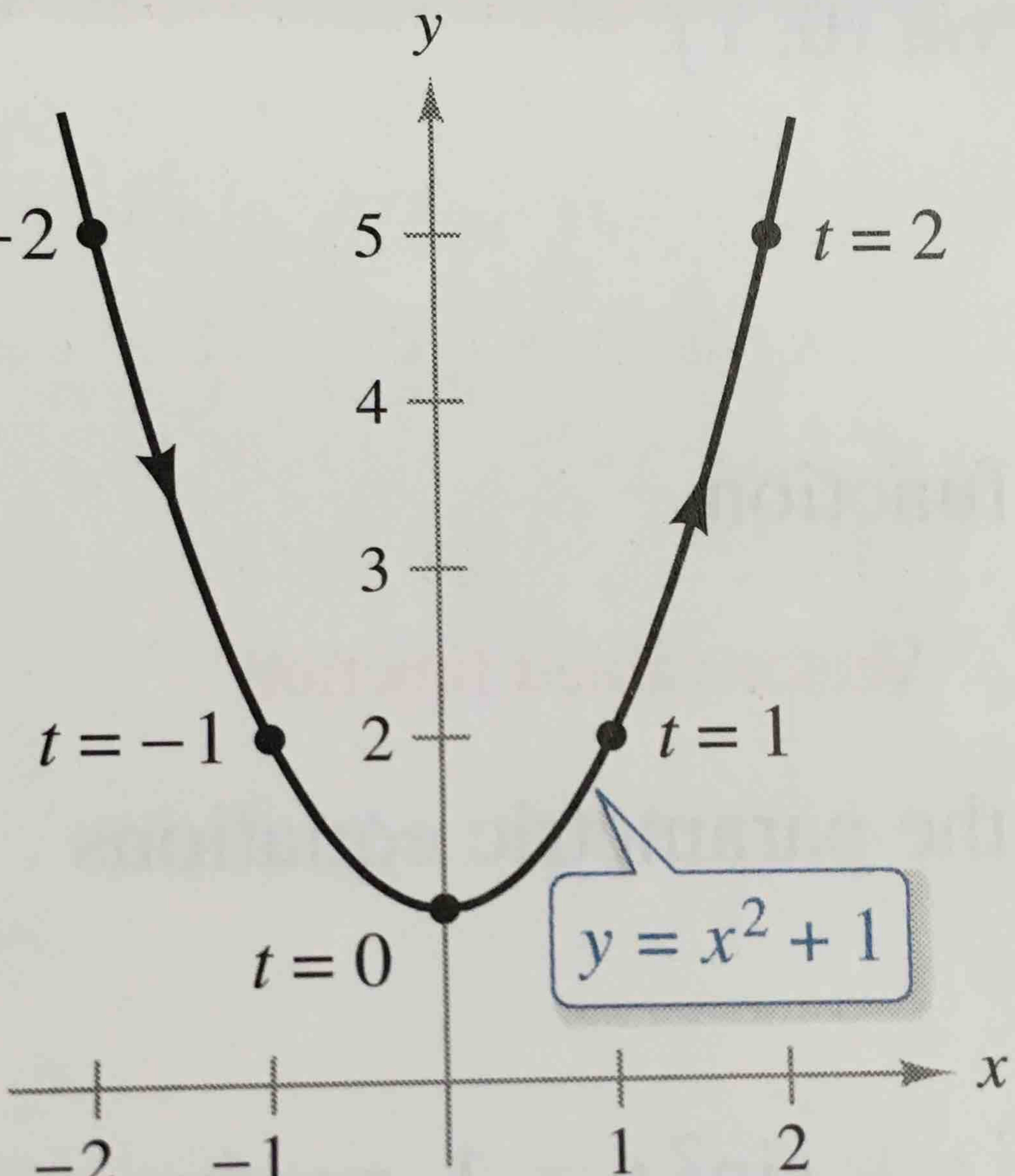
In 1953, Francis Crick and James D. Watson discovered the double helix structure of DNA.

EXAMPLE 3 Representing a Graph: Vector-Valued Function

Represent the parabola

$$y = x^2 + 1$$

by a vector-valued function.



There are many ways to parametrize this graph. One way is to let $x = t$.

Figure 12.4

EXAMPLE 4 Representing a Graph: Vector-Valued Function

Sketch the space curve C represented by the intersection of the semiellipsoid

$$\frac{x^2}{12} + \frac{y^2}{24} + \frac{z^2}{4} = 1, \quad z \geq 0$$

and the parabolic cylinder $y = x^2$. Then find a vector-valued function to represent the graph.

The curve C is the intersection of the semiellipsoid and the parabolic cylinder.

Figure 12.5

Definition of the Limit of a Vector-Valued Function

1. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

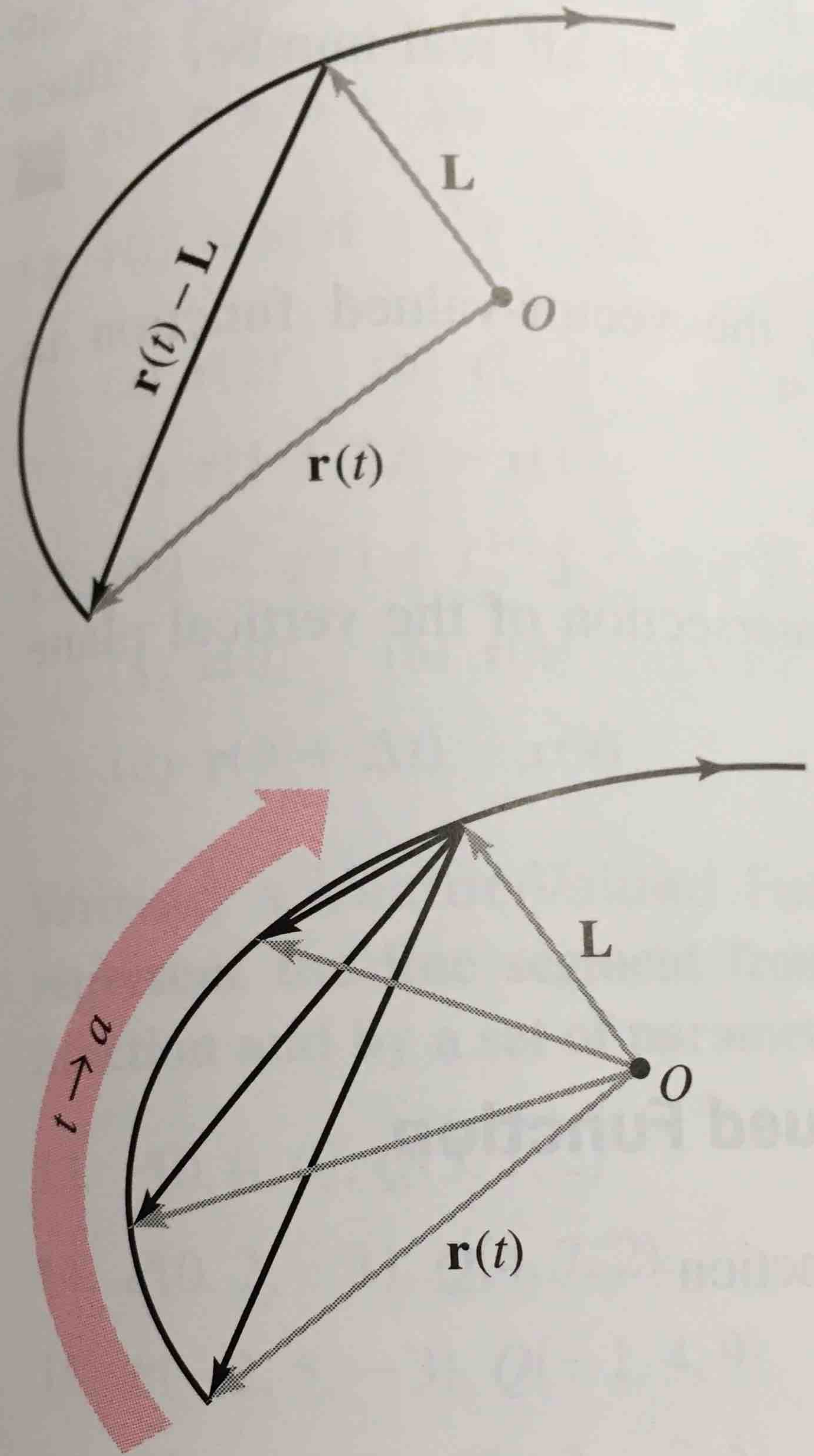
$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j} \qquad \text{Plane}$$

provided f and g have limits as $t \to a$.

2. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j} + \left[\lim_{t \to a} h(t)\right]\mathbf{k} \qquad \text{Space}$$

provided f, g, and h have limits as $t \to a$.



As t approaches a, $\mathbf{r}(t)$ approaches
limit $\mathbf{L}$. For the limit $\mathbf{L}$ to exist, it
not necessary that $\mathbf{r}(a)$ be defined
that $\mathbf{r}(a)$ be equal to $\mathbf{L}$.

Figure 12.6

Definition of Continuity of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ is **continuous at the point** given by $t = a$ when the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

A vector-valued function $\mathbf{r}$ is **continuous on an interval** I when it is continuous at every point in the interval.

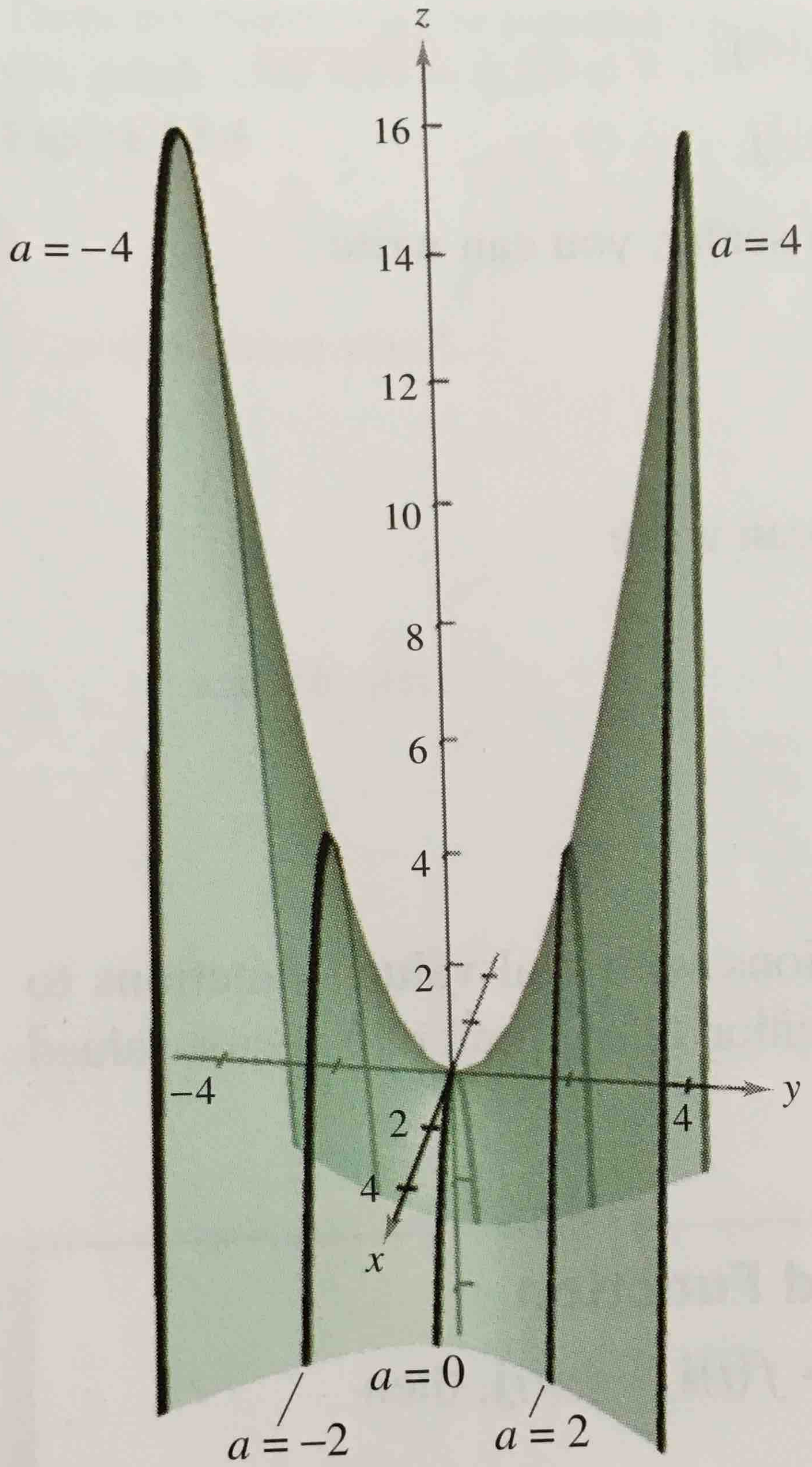
EXAMPLE 5 **Continuity of a Vector-Valued Function**

Discuss the continuity of the vector-valued function

$a = 4$

$$\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k}$$ *a* is a constant.

at $t = 0$.



For each value of a, the curve represented by the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k}$ is a parabola.

Figure 12.7

EXAMPLE 6 Continuity of a Vector-Valued Function

Determine the interval(s) on which the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + \sqrt{t+1}\,\mathbf{j} + (t^2 + 1)\mathbf{k}$$

is continuous.