Greens Theorem
15.4

George Green: Self educated published paper 1820's

First Concept: <u>Simply Connected Plane Region</u>:

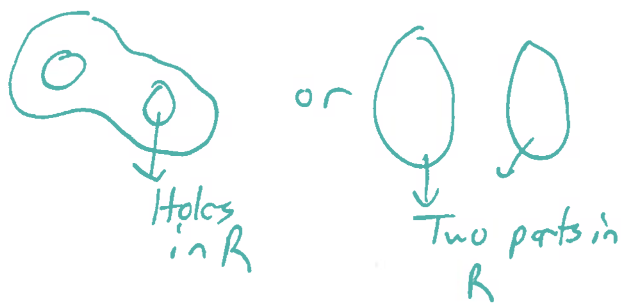
A curve (C) which does not cross itself when traced by a vector function $r(t) = x(t)i + y(t)j$ is called <u>Simple</u>

Ex

Simply connected

Simply connected has no separate Parts or holes in the Region

Not Simply Connected



Theorem 15.8

$$\int_C M\,dx + N\,dy = \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

IOW: The value of the double integral over the simply connected region R is determined by the value of the line integral around R

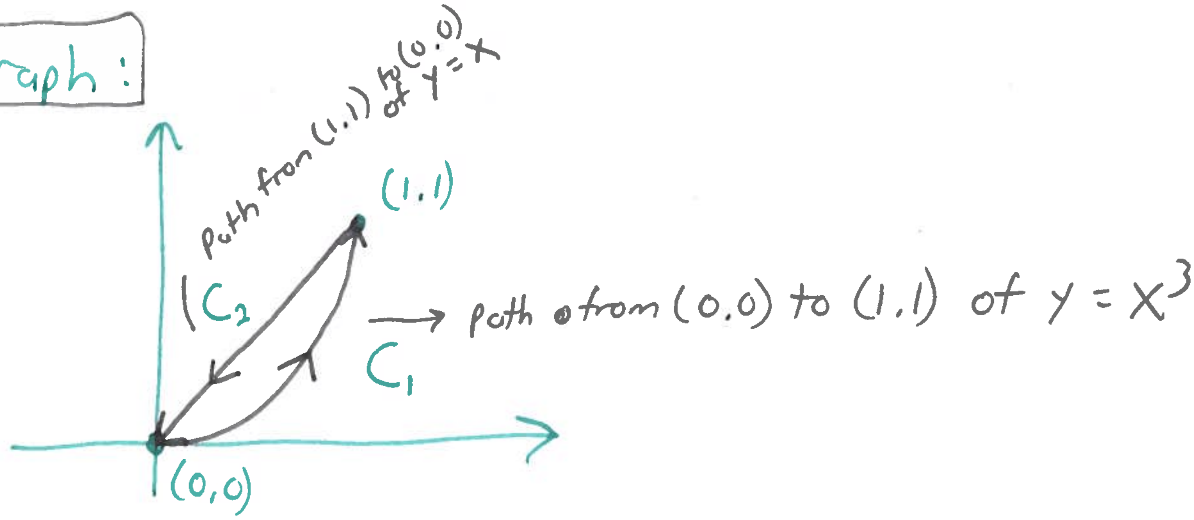
Often use a circle to indicate line integral

$$\oint_C \quad \text{or} \quad \oint_C$$

Ex 1: $\displaystyle\int_C y^3\,dx + (x^3+3xy^2)\,dy$

where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along graph of $y = x$

$\boxed{\text{Graph:}}$



Labeling: $M = y^3$ and $N = x^3 + 3xy^2$

Find Partial Derivatives:

$$\frac{\partial N}{\partial x} = 3x^2 + 3y^2$$

$$\frac{\partial M}{\partial y} = 3y^2$$

Apply Greens Theorem

$$\boxed{\int_R\int\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA}$$

$$=\int_0^1\int_{x^3}^{x}\left[3x^2+3y^2\right]-3y^2\Big]dy\,dx$$

Set up bands for
dy dx

dx from $0\leq x\leq 1$
dy from $x^3\leq y\leq x$

$$=\int_0^1\int_{x^3}^{x}\left[3x^2\right]dy\,dx$$

$$=\int_0^1 3x^2y\Big]_{x^3}^{x}dx$$

$3x^2(x)-3x^2(x^3)$
$3x^3-3x^5$

$$=\int_0^1\left[3x^3-3x^5\right]dx$$

$$=\frac{3x^4}{4}-\frac{3x^6}{6}\Big]_0^1$$

$$=\boxed{\frac{1}{4}}$$

This theorem is a time save for calculating line integrals (when conditions are met)